Dérivation des équations de Schrödinger non linéaires par une méthode des caractéristiques en dimension infinie

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Dérivation des équations de Schrödinger non-linéaires par une méthode des caractéristiques en dimension infinie

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Chapter 1

Introduction

1.1 Cadre général

Le sujet de cette thèse est la dérivation de la limite de champ moyen par une méthode des caractéristiques en dimension infinie. L'approximation de champ moyen est un concept bien connu en physique, effectif dans de nombreux modèles. Considérons pour commencer un nombre N de particules quantiques identiques, dans l'espace de configuration $\mathbb{R}^d$, interagissant deux-à-deux à travers un potentiel scalaire W. La mécanique quantique est le cadre adéquat pour décrire un tel système par le biais des fonctions d'onde $\Psi^{(N)} \in L^2(\mathbb{R}^{dN})$ de norme un, ou plus généralement des états à trace, et des observables. Dans la suite on adoptera la notation 'bra-ket', utilisée par les physiciens (i.e. $|\Psi^{(N)}\rangle$ ou $\langle\Psi^{(N)}|$). Ainsi la mesure d'une grandeur physique correspond à l'évaluation d'un observable A dans l'état $\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|$ (projecteur orthogonal engendré par $\Psi^{(N)}$), i.e.,

$$\langle A\rangle_{\Psi^{(N)}} = \langle\Psi^{(N)}, A\Psi^{(N)}\rangle_{L^2(\mathbb{R}^{dN})} = \operatorname{Tr}\,[A\varrho_N]. \tag{1.1.1}$$

Le système de N particules est décrit par le Hamiltonien adimensionné suivant:

$$H_N = \sum_{i=1}^{N} A_i + \frac{\mathbf{1}}{\mathbf{N}} \sum_{1\leqslant i<j\leqslant N} W_{i,j}. \tag{1.1.2}$$

Ainsi, l'Hamiltonien H_N se décompose en une partie énergie cinétique H_N^0 :

$$H_N^0 = \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} \mathrm{Id} \otimes\cdots\otimes \underbrace{A}_{\text{i}^{\text{ème}}\text{ position}} \otimes\cdots\otimes \mathrm{Id}\,, \tag{1.1.3}$$

et un potentiel W_N donné par la somme des interactions entre les particules i et j :

$$W_N = \frac{\mathbf{1}}{\mathbf{N}} \sum_{1\leqslant i<j\leqslant N} W_{i,j}. \tag{1.1.4}$$

La constante de couplage $\frac{\mathbf{1}}{\mathbf{N}}$ correspond à l'échelle propre au champ moyen. En effet, dans ce cas, l'interaction est proportionnelle à la partie énergie cinétique quand le nombre de particules N grandit.

L'opérateur autoadjoint A agit sur un domaine $\mathcal{D} \subset L^2(\mathbb{R}^d)$, alors que le potentiel $W_{i,j}$ agit sur un espace $\mathcal{D}_2 \subset L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \simeq L^2(\mathbb{R}^{2d})$. En pratique, dans le cadre des particules non relativistes, l'opérateur A est de la forme $A = -\Delta + U$ avec U une fonction mesurable. Les particules libres, confinées dans un piège ou sous influence d'un champ magnétique couplé avec un champ électrique feront également l'objet d'une étude dans cette thèse. D'un autre coté, le potentiel $W_{i,j}$ est de la forme $W(x_i - x_j)$ où $x_i, x_j \in \mathbb{R}^3$ jouent le rôle des positions des particules i et j avec W un potentiel singulier remplissant les conditions suffisantes d'existence et d'unicité globale pour une équation classique de type (**Hartree**).

1.1.1 Dynamique de champ moyen

Le système est décrit à l'instant t par une fonction d'onde $\Psi^{(N)}(t) \in L^2(\mathbb{R}^{dN})$ selon l'équation de Schrödinger

$$\begin{cases} i\partial_t \Psi^{(N)}(t) = H_N \Psi^{(N)}(t) \\ \Psi^{(N)}(0) = \Psi^{(N)}. \end{cases} \qquad \textbf{(Schrödinger à N corps)}$$

Ainsi cette équation linéaire admettra une solution sous la condition d'une réalisation autoadjointe de l'opérateur H_N (1.1.2). Un des premiers résultats dans ce sens se trouve dans le travail de Kato dans les années 1950 où il considera une perturbation de type multiplication $W_{i,j} = W(x_i - x_j)$ avec W satisfaisant les hypothèses du théorème de Kato-Reillich [100, Théorème X.12] et l'opérateur $A = -\Delta$. L'interaction W_N étant une perturbation d'un hamiltonien libre, l'approche par le [100, Théorème X.17] a permis d'étendre l'existence de solutions à l'équation (**Schrödinger à N corps**) pour des formes quadratiques relativement bornées par rapport à l'hamiltonien libre H_N^0 (1.1.3). Ainsi la réalisation autoadjointe de H_N permet de définir la solution $\Psi^{(N)}(t) = e^{-itH_N}\Psi^{(N)}$. En poursuivant l'analogie avec les opérateurs à trace, l'évolution d'états quantiques $\varrho_N(t)$ est donnée par l'équation de Von Neumann

$$\begin{cases} i\partial_t \varrho_N(t) = [H_N, \varrho_N] = H_N \varrho_N - \varrho_N H_N \\ \varrho_N(0) = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|. \end{cases} \qquad \textbf{(Von Neumann)}$$

Dans cette thèse nous nous intéresserons aux particules de spin entier qui obéissent à la statistique de Bose-Einstein: les bosons. L'étude des particules bosoniques est motivée par de nombreux phénomènes quantiques tels les transitions de phase dans le cadre de la superfluidité, les condensats de Bose-Einstein pour des atomes ultra-froids ou encore l'étude de vortex en supraconductivité. Dans la nature, le photon, les gluons ou encore les particules responsables de l'interaction faible (Z^0, W^-, W^+) sont des bosons. Mathématiquement, la statistique de Bose-Einstein se traduit pour les bosons par une symétrie dans leur fonction d'onde. En effet pour $x_1, ..., x_N \in \mathbb{R}^d$ la fonction d'onde $\Psi^{(N)}(x_1, ..., x_N)$ est inchangée après permutation des variables, i.e:

$$\Psi^{(N)}(x_1, ..., x_N) = \Psi^{(N)}(x_{\sigma_1}, ..., x_{\sigma_N}),\ \forall \sigma \in \Sigma_N, \tag{1.1.5}$$

où Σ_N désigne le groupe symétrique à N éléments. Par la suite on s'intéressera au sous espace des fonctions symétriques $L_s^2(\mathbb{R}^{dN}) := \mathcal{S}_N(L^2(\mathbb{R}^{dN}))$ où $\mathcal{S}_N$ désigne la projection orthogonale

$$\mathcal{S}_N \Psi^{(N)}(x_1, \cdots, x_N) = \frac{1}{N!} \sum_{\sigma \in \Sigma_N} \Psi^{(N)}(x_{\sigma_1}, \cdots, x_{\sigma_N}).$$

Revenons à présent à notre système de N bosons. L'approximation de champ moyen consiste à considérer qu'une particule évolue dans un champ moyen généré par l'ensemble des autres particules. Pour un système de bosons dont l'évolution est gouvernée par l'équation (**Schrödinger à N corps**), l'essentiel des corrélations entre les particules se trouve dans l'état initial. En effet, en considérant des états quantiques décorellés (états factorisés) de la forme $\Psi^{(N)} = \Psi^{\otimes N}$ avec $\Psi \in L^2(\mathbb{R}^d)$ l'approximation de champ moyen se traduit par

$$e^{-itH_N}\Psi^{(N)} \approx \Psi_t^{\otimes N},\ N \text{ grand}, \tag{1.1.6}$$

avec $\Psi_t \in L^2(\mathbb{R}^d)$ une fonction qui satisfait l'équation de champ moyen classique, aussi appelée équation de Hartree,

$$\begin{cases} i\partial_t z_t = \partial_{\bar{z}} h(z_t, \bar{z}_t) \\ z_{|t=0} = \Psi\,, \end{cases} \tag{\textbf{Hartree Générale}}$$

avec

$$h(z,\bar{z}) = \langle z, Az\rangle_{L^2(\mathbb{R}^d)} + \frac{1}{2}\langle z^{\otimes 2}, W_{1,2}\, z^{\otimes 2}\rangle_{L^2(\mathbb{R}^d)}, \tag{\textbf{Energie de Hartree}}$$

qui est appelée énergie de champ moyen (ou énergie de Hartree). Dans le cadre du Hamiltonien libre à une particule $A = -\Delta$ et d'une interaction $W_{i,j} = W(x_i - x_j)$, l'équation de Hartree s'écrit simplement

$$\begin{cases} i\partial_t z_t = Az_t + W * |z_t|^2 z_t \\ z_{|t=0} = \Psi. \end{cases} \tag{\textbf{Hartree}}$$

La non-linéarité dans ces équations (**Hartree Générale**)-(**Hartree**) traduit l'idée d'interaction moyenne quand le nombre de particules est grand. Expérimentalement, l'approximation de champ moyen devient souvent effective dès que le nombre de particules N dépasse quelques dizaines. L'approximation (1.1.6) traduit l'idée que les corrélations entre les particules existent mais qu'elles sont négligeables quand le nombre de particules est grand. Mais quel sens mathématique donné à l'expression (1.1.6)? En mécanique quantique les quantités introduites en (1.1.1) vont permettre de donner un sens à la dynamique de champ moyen.

Historique des résultats sur la dynamique de champ moyen

Dans cette partie, on abordera les différents résultats obtenus depuis les années 1970. La validation mathématique de cette approximation a commencé par l'étude d'états quantiques bien précis (états décorellés, états cohérents). La plupart des résultats concerne le cas de l'opérateur à une particule $A = -\Delta + U$ et une interaction $W_{i,j} = W(x_i - x_j)$ où W est une fonction dont la régularité sera discutée et U un potentiel soit confinant soit une perturbation du Laplacien. Le cadre de bosons non-relativistes sous influence d'un champ magnétique couplé avec un champ électrique a fait l'objet d'une étude plus récente que l'on commentera. La vraie question qui a motivé les travaux d'un point de vue mathématique est la validation de la limite de champ moyen pour des potentiels coulombiens. Cela fut effectuée plus récemment dans les années 2000.

La dynamique de champ moyen a commencé par une étude de Hepp dans [63], par la méthode susnommée. Les états quantiques considérés sont dits cohérents (on donnera une définition précise de cette notion par la suite) et l'approximation de champ moyen est effective pour des potentiels U et

W très réguliers. Ensuite, le travail de Ginibre et Vélo [52] a permis d'étendre de façon significative l'approximation pour des potentiels singuliers W ayant une singularité locale et une décroissance contrôlée à l'infini pour de grandes dimensions ($n \geqslant 5$) et de type $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ avec $p > \frac{d}{2}$. Une condition supplémentaire est requise sur $W^- = \max\{-W, 0\}$ pour assurer la conservation de l'énergie et le caractère globalement bien posé de l'équation (**Hartree**). Cependant cette extension proposée par Ginibre et Vélo pour des potentiels de ce type ne couvre que des états dits cohérents et utilise fortement la structure de l'espace de Fock et la seconde quantification. Ainsi, cela n'inclut pas les états initiaux avec un nombre fixé de particules $\Psi^{\otimes N}$.
En 1980, une autre approche introduite par Spohn dans [110] a permis de prouver un théorème de convergence pour des états quantiques décorellés sous la condition que A et $W_{i,j}$ soient des opérateurs bornés autoadjoints. Introduisons la notation $\text{Tr}_{[n,N]}$ qui définit les traces partielles indexées par $n, n+1, ..., N$ et Tr_n qui désigne la trace sur le n-ième espace de Hilbert $L^2(\mathbb{R}^{dn})$.

Theorem 1.1.1. *Soit $\varrho_N = \varrho^{\otimes N}$ un état quantique normalisé avec $\varrho \in \mathcal{L}(L^2(\mathbb{R}^d))$ et $\text{Tr}[\varrho] = 1$. Alors*

$$\lim_{N \to +\infty} \text{Tr}_{[n+1,N]}[e^{-itH_N} \varrho_N e^{itH_N}] = \varrho(t)^{\otimes n},$$

en norme trace sur les applications bornées de $L^2(\mathbb{R}^{dN})$. L'opérateur $\varrho(t)$ satisfait l'équation de Hartree au sens de Von Neumann, i.e

$$i\partial_t \varrho(t) = [A, \varrho(t)] + \text{Tr}_2[W_{1,2} + W_{2,1}, \varrho(t) \otimes \varrho(t)]. \tag{1.1.7}$$

La méthode utilisée par Spohn consiste à écrire une série perturbative pour la quantité

$$\text{Tr}_{[n+1,N]}(e^{-itH_N} \varrho_N e^{itH_N}),$$

car celle-ci vérifie une équation différentielle. Ainsi, de fortes conditions du théorème 1.1.1 sur le potentiel W sont requises pour faire converger la série.
Bardos, Golse et Mauser dans [18] ont ensuite prolongé l'idée de Spohn en utilisant les matrices à densité réduite et les hiérarchies (**BBGKY**) (Bogoliubov, Born, Green, Kirkwood, Yvon). Introduisons brièvement ce vocabulaire. Considérons un opérateur A borné n'agissant que sur un nombre fixé k de variables, i.e $A \in \mathcal{L}(L^2(\mathbb{R}^{dk}))$, où $\mathcal{L}(L^2(\mathbb{R}^{dk}))$ désigne les opérateurs bornés de $L^2(\mathbb{R}^{dk})$ dans $L^2(\mathbb{R}^{dk})$. En considérant un état quantique normalisé $\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|$, on définit les matrices à densité réduite grâce à leurs noyaux

$$\gamma_{k,N}(x_1, ..., x_k; y_1, ..., y_k) = \int_{\mathbb{R}^{d(N-k)}} \Psi^{(N)}(x_1, ..., x_k; z)\overline{\Psi^{(N)}(y_1, ..., y_k; z)}\, dz. \tag{1.1.8}$$

Ainsi défini le noyau est symétrique par rapport aux permutations des variables $(x_1, ..., x_k)$ ou $(y_1, ..., y_k)$. Cela induit ainsi un opérateur sur le sous espace des fonctions symétriques $L^2_s(\mathbb{R}^{dk})$ de trace égale à un, positif que l'on note $\gamma_{k,N}$. Ainsi, l'évolution d'états quantiques du type $|e^{-itH_N}\Psi^{\otimes N}\rangle\langle e^{-itH_N}\Psi^{\otimes N}|$ permet de définir une famille de matrices à densité réduite $(\gamma^t_{k,N})_{t\in\mathbb{R}}$. Dans [19, 50] l'approximation de champ moyen est présentée sous la forme du théorème suivant.

Theorem 1.1.2. *Soit $\Psi \in H^2(\mathbb{R}^3)$ et $\Psi(t) \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}^3))$, une solution globale continue de l'équation de* (**Hartree**) *de régularité $H^2(\mathbb{R}^3)$ avec pour donnée initiale Ψ. Soit $\Psi^{(N)}$ une solution de l'équation*

(**Schrödinger à N corps**) *vérifiant au temps initial* $\Psi^{(N)}(0) = \Psi^{\otimes N}$. *Alors pour tout* $k \geqslant 1$*, pour tout réel* $t \in \mathbb{R}$ *et pour tout opérateur compact* $B \in \mathcal{L}(L^2(\mathbb{R}^{3k}))$*,*

$$\lim_{N\to+\infty} \mathrm{Tr}\,[B\,\gamma_{k,N}^t] = \langle \Psi(t)^{\otimes k}, B\,\Psi(t)^{\otimes k}\rangle_{L^2(\mathbb{R}^3)}\,. \tag{1.1.9}$$

La convergence (1.1.9) correspond à la convergence faible étoile dans l'espace des opérateurs à trace. La démonstration de ce théorème s'effectue en plusieurs étapes. La première consiste à établir une équation différentielle sur les noyaux $\gamma_{k,N}^t(X,Y)$ avec $X \in \mathbb{R}^k,\ Y \in \mathbb{R}^k$ en utilisant l'équation de (**Von Neumann**) et on obtient

$$\begin{aligned} i\partial_t\gamma_{k,N}^t(X,Y) &= -(\Delta_X - \Delta_Y)\gamma_{k,N}^t(X,Y) + \tfrac{1}{N}\textstyle\sum_{1\leqslant j<l\leqslant k}[W(x_j-x_l) - W(y_j-y_l)]\gamma_{k,N}^t(X,Y) \\ &+ \tfrac{N-k}{N}\textstyle\sum_{1\leqslant j\leqslant k}\int_{\mathbb{R}^3}[W(x_j-z) - W(y_k-z)]\gamma_{k+1,N}^t(X,z,Y,z)\,dz,\ 1\leqslant k\leqslant N-1. \end{aligned} \tag{\textbf{BBGKY}}$$

La seconde étape est le passage à la limite dans (**BBGKY**). Pour se faire, la compacité est nécessaire pour obtenir au moins l'existence d'une hiérarchie limite. Dans [19] l'existence d'une hiérarchie limite est obtenue par un potentiel $W \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ borné inférieurement. Ainsi, une limite possible $(\gamma_k^t)_{k\geqslant 1}$ satisfait l'équation différentielle pour $k \geqslant 1$

$$i\partial_t\gamma_k^t(X,Y) = -(\Delta_X - \Delta_Y)\gamma_k^t(X,Y) + \sum_{1\leqslant j\leqslant k}\int_{\mathbb{R}^3}[W(x_j-z) - W(y_k-z)]\gamma_{k+1}^t(X,z,Y,z)\,dz. \tag{\textbf{Limite BBGKY}}$$

Cependant, l'existence est prouvée dans [19] mais l'unicité nécessite une hypothèse plus forte sur le potentiel W (W borné). Cela généralise néanmoins le travail de Spohn puisque cela inclut le potentiel de Coulomb répulsif pour la convergence mais pas pour l'unicité. En 2001, Erdös et Yau ont démontré l'unicité pour cette hiérarchie limite (**Limite BBGKY**) pour un potentiel coulombien en supposant plus de régularité sur les états initiaux (typiquement un état initial Ψ dans un espace de Sobolev adapté). Ainsi, la méthode basée sur les hiérarchies (**BBGKY**) permet d'obtenir des potentiels singuliers (Coulombiens) mais ne donne pas d'information sur un éventuel taux de convergence. Cependant, en utilisant la formulation de Duhamel dans l'expression (**BBGKY**) et en écrivant une série perturbative dans l'intégrande, on peut comparer la limite γ_k^t avec les projecteurs orthogonaux $|\Psi(t)\rangle\langle\Psi(t)|^{\otimes k}$ et obtenir localement en temps, l'existence de constantes $C > 0$ et $t_0 > 0$ telle que

$$\mathrm{Tr}\ \left[\left|\gamma_{k,N}^t - |\Psi(t)\rangle\langle\Psi(t)|^{\otimes k}\right|\right] \leqslant \frac{C^k}{N},\ \forall |t| \leqslant t_0.$$

Rodnianski et Schlein ont amélioré ce résultat dans [103] en utilisant la méthode de Hepp et les outils de la seconde quantification pour des potentiels W satisfaisant $W(1-\Delta)^{-\frac{1}{2}} \in \mathcal{L}(L^2(\mathbb{R}^d))$. Le taux de convergence étant donné par l'inégalité pour tout $k \geqslant 1$

$$\exists C(k), K(k) > 0,\ \mathrm{Tr}\ \left[\left|\gamma_{k,N}^t - |\Psi(t)\rangle\langle\Psi(t)|^{\otimes k}\right|\right] \leqslant \frac{C(k)}{\sqrt{N}}e^{K(k)t},\ \forall t \in \mathbb{R}.$$

Les mêmes auteurs ont également obtenu un taux de convergence pour les états cohérents dans [103]. Une autre méthode a été développée par Fröhlich, Graffi et Knowles dans [47] (voir aussi [48]). Elle est basée sur un théorème d'Egorov. Ce théorème en dimension finie permet de suivre l'évolution d'observables et l'utilisation de la quantification comme le caractère "réciproque" de la limite semi-classique $\hbar \to 0$. L'idée ainsi est de voir la "réciproque" de la limite de champ moyen comme étant la seconde quantification introduite par Dirac. Le paramètre semiclassique n'est plus $\hbar$ mais l'inverse du nombre des particules $\frac{1}{N}$. Ainsi obtenir un théorème similaire à celui d'Egorov pour la limite de champ moyen nécessite la prise en compte d'états particuliers, les états factorisés, et d' exploiter les symétries des fonctions d'onde. Le théorème suivant résume cette approche et ne nécessite aucune régularité pour les états initiaux.

Theorem 1.1.3. *Considérons le problème à N corps pour un opérateur $A = -\Delta$ et une interaction $W_{i,j} = W(x_i - x_j)$ avec $W \in L^\infty(\mathbb{R}^d)$. Notons A_N^p l'opérateur donné par la formule suivante pour $\Psi^{(N)}(x_1, \cdots, x_N) \in L_s^2(\mathbb{R}^{dN})$ et a^p un opérateur borné de $L^2(\mathbb{R}^{dp})$ dans $L^2(\mathbb{R}^{dp})$*

$$(A_N^p \Psi^{(N)})(x_1, \cdots, x_N) = \frac{N(N-1)\cdots(N-p+1)}{N^p} \big(\mathcal{S}_p a^p \otimes \ \mathrm{Id}^{\ \otimes(N-p)} \mathcal{S}_p \Psi^{(N)}\big)(x_1, \cdots, x_N).$$

On note Ψ_t la solution de l'équation de (**Hartree**)*. Soit $\Psi^{(N)}(0) = \Psi^{\otimes N} = \Psi(x_1)\cdots\Psi(x_N)$ un état quantique normalisé alors on a l'égalité suivante pour tout $t \in \mathbb{R}$,*

$$\lim_{N\to\infty} \langle \Psi^{(N)}(0), e^{itH_N} A_N^p e^{-itH_N} \Psi^{(N)}(0)\rangle = \langle \Psi_{p,t}\, , \, a^p\, \Psi_{p,t}\rangle =: a^p(\Psi_t), \tag{1.1.10}$$

avec $\Psi_{p,t}(x_1, \cdots, x_N) := \Psi_t(x_1)\cdots\Psi_t(x_p)$.

Pour obtenir cette convergence (1.1.10), l'idée est d'étudier l'évolution des observables dans l'espace de Fock (espace permettant de traiter un nombre arbitraire de particules (voir Section 2) et d'obtenir une égalité au sens faible

$$e^{-itH_N} A_N^p e^{itH_N} = a^p(\Psi_t)) + R_N(t),\ \forall p \geqslant 1,\ \forall N \in \mathbb{N}^*. \tag{1.1.11}$$

Le reste $R_N(t)$ se contrôle avec les extensions de Schwinger-Dyson et un contrôle uniforme en N (comptage de boucles des graphes de Feynman où chaque boucle contient une contribution d'ordre $\frac{1}{N}$). La régularité se situe essentiellement dans le caractère borné de l'opérateur a^p. Dans [48] les auteurs étendent la convergence (1.1.10) au cas de potentiels $W \in L^3_{\mathrm{w}}(\mathbb{R}^3) + L^\infty(\mathbb{R}^d)$.
En 2009, dans [95], Pickl propose une nouvelle méthode permettant de traiter la limite de champ moyen sans utiliser les hiérarchies (**BBGKY**) mais plutôt des estimations de type Gronwall sur les quantités $1 - |\Psi(t)\rangle\langle\Psi(t)|$ avec Ψ solution de l'équation (5.2.10). Cette idée a permis en 2012 d'obtenir un résultat d'approximation de champ moyen dans le cadre non relativiste $A = -\Delta$ pour des potentiels critiques en dimension 3 ($W = \frac{1}{|x|^2}$) et sous-critique ($W = \frac{1}{|x|^\alpha}$, $\alpha < 2$). Cela a également fourni un taux de convergence pour les matrices à densité réduite dans [69]. La méthode adaptée à l'étude des états factorisés permet aussi de traiter le cadre semi-relativiste $A = \sqrt{-\Delta + m^2}$ avec une interaction critique $W = \frac{1}{|x|}$. Ce dernier cadre a fait l'objet d'un travail d'Elgart et Schlein [39] utilisant les équations (**BBGKY**) et de la régularité pour les états quantiques initiaux. La combinaison des méthodes introduites par Pickl, Knowles, et des hiérarchies (**BBGKY**) utilisées par Spohn a permis à Luhrmann dans [88] d'obtenir un résultat d'approximation de champ moyen et un taux de convergence dans le sens développé par le théorème 1.1.2 pour des bosons non-relativistes sous influence d'un champ magnétique $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$ couplé avec un champ électrique V. Le Hamiltonien à N corps dans ce cadre est donné par

$$H_N = \sum_{j=1}^N \big[(\nabla_{x_j} - i\mathcal{A}(x_j))^2 + V(x_j)\big] + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} W(x_i - x_j).$$

Les hypothèses permettant cette convergence sont faibles au niveau de l'interaction $W(x_i - x_j)$ où W est une fonction paire mesurable de type $W = \frac{\lambda}{|x|}$ ou $W \in L^\infty(\mathbb{R}^3)$. Cependant le champ magnétique $\mathcal{A}$ et le champ électrique V ont la régularité suffisante afin d'utiliser des estimations de Strichartz magnétiques.
L'évolution des états cohérents a également été étudiée dans le cadre d'interactions plus singulières dans [6]. Ammari et Breteaux ont prouvé l'approximation de champ moyen, appelée dans ce cadre "propagation du chaos", en utilisant les travaux de Ginibre, Vélo [52] et Schlein et Rodnianski dans [103] pour un potentiel $W = \delta$, en dimension un. Approfondissons à présent le cadre de travail de la seconde quantification pour traiter le problème de la limite de champ moyen. La stratégie développée dans les travaux récents d'Ammari et Nier [9, 10, 11, 12] puis Ammari et Falconi dans [13] est de conserver l'esprit d'un problème semiclassique, c'est-à-dire de ne plus regarder l'évolution d'observables (voir la discussion sur le théorème d'Egorov), mais plutôt de considérer la dynamique des états quantiques. Cette remarque se résume par l'égalité suivante. Soit $\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|$ un état quantique normalisé et A un observable quantique. Alors

$$\text{Tr}\,[\varrho_N(t)A] = \text{Tr}\,[\underbrace{e^{-itH_N}\varrho_N e^{itH_N}}_{\text{Evolution des états}} A] = \langle\Psi^{(N)}, \underbrace{e^{itH_N} A e^{-itH_N}}_{\text{Evolution des observables}} \Psi^{(N)}\rangle. \tag{1.1.12}$$

Ainsi, la régularité nécessaire dans le passage à la limite va reposer sur des hypothèses sur les états quantiques initiaux (états avec masse et énergie finies). La représentation de Fock va permettre de traiter le problème dynamique et aussi variationnel. Soient a et a^*, respectivement les opérateurs d'annihilation et de création (voir définition précise dans la Section 2), l'énergie du système est décrite par le Hamiltonien H_ε

$$\begin{aligned} H_\varepsilon &= \varepsilon \int_{\mathbb{R}^d} \nabla a^*(x)\nabla a(x)\,dx + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^{2d}} a^*(x)a^*(y)W(x-y)a(x)a(y)\,dxdy \qquad (1.1.13)\\ &= \underbrace{\int_{\mathbb{R}^d} \nabla a_\varepsilon^*(x)\nabla a_\varepsilon(x)\,dx}_{H_\varepsilon^0:\text{partie énergie cinétique}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^{2d}} a_\varepsilon^*(x)a_\varepsilon^*(y)W(x-y)a_\varepsilon(x)a_\varepsilon(y)\,dx\,dy}_{W_\varepsilon:\text{interaction}}, \end{aligned}$$

où ε va jouer le rôle du paramètre semiclassique tendant vers 0 quand le nombre de particules est grand. Ainsi a_ε^* et a_ε seront d'ordre $\sqrt{\varepsilon}$ et vérifient les relations de commutation canonique

$$[a_\varepsilon(x), a_\varepsilon^*(y)] = \varepsilon\,\delta(x-y),\ [a_\varepsilon^*(x), a_\varepsilon^*(y)] = [a_\varepsilon(x), a_\varepsilon(y)] = 0. \tag{CCR}$$

Le changement d'échelle (qui conserve le nombre de particules) se traduit par

$$[\varepsilon^{-1}H_\varepsilon]_{|L_s^2(\mathbb{R}^{dN})} = H_N,\ \text{avec}\, \varepsilon = \frac{1}{N},$$

où H_N est le Hamiltonien à N corps introduit précédemment dans le cadre $W_{i,j} = W(x_i - x_j)$. Ainsi sous la forme (1.1.13) on peut "quantifier" au moins formellement cette énergie grâce à ce que l'on appellera la quantification de Wick ou ordre de Wick que l'on définira dans la Section 2.

$$h(z,\bar{z})^{Wick} = H_\varepsilon,$$

avec $h(z, \bar{z})$ l'énergie classique introduite en (**Energie de Hartree**). En dimension infinie la difficulté réside dans le bon choix des classes de symboles associées à des observables opérant dans l'espace de Fock. Deux types de résultats vont ainsi émerger du travail [9, 10, 11, 12]. La perte de compacité due à la dimension infinie sera compensée par la régularité des symboles de Wick (symboles à noyaux compacts) ou par des symboles ne dépendant que d'un nombre fini de variables dans le cas de la quantification de Weyl. Une hypothèse de régularité quantique sur les états, appellée condition (**PI**) et rappelée dans la Section 2.2.3 permet également de contourner la perte de compacité. Dans ce contexte, la méthode développée dans [12] repose sur un outil d'analyse semiclassique, les mesures de Wigner. Elles joueront un rôle dans la dérivation de la limite de champ moyen en permettant d'établir un lien direct entre les quantités classiques et quantiques (estimations a priori, localisation des états). Nous aborderons en détails la stratégie développée par Ammari et Nier et cette thèse permettra de compléter les résultats obtenus dans [11, 12]. La prochaine définition introduit les mesures de Wigner associées à des états quantiques normalisés.

Definition 1.1.4. *Soit une famille d'états normaux* $\{\varrho_N := |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$*. L'espace* $\mathcal{M}(\varrho_N,\, N \in \mathbb{N}^*)$ *des mesures de Wigner associées à la famille* $(\varrho_N)_{N\in\mathbb{N}}$ *est l'espace des mesures de probabilité borélienne* μ *sur* $L^2(\mathbb{R}^d)$ *tel qu'il existe une sous-suite* $(N_k)_{k\in\mathbb{N}}$ *vérifiant:*

$$\forall \xi \in L^2(\mathbb{R}^d),\ \lim_{k\to\infty} \langle \Psi^{(N_k)}, \mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N_k)}\rangle = \int_{L^2(\mathbb{R}^d)} e^{2i\pi\,\mathrm{Re}\,\langle \xi, z\rangle}\, d\mu(z),$$

où $\mathcal{W}(\sqrt{2}\pi\xi)$ *désigne le groupe unitaire de Weyl généré par l'opérateur de champ*

$$\Phi(\xi) = \frac{1}{\sqrt{2}}(a_\varepsilon(\xi) + a_\varepsilon^*(\xi)) \quad \textit{avec} \quad \varepsilon = \frac{1}{N_k}\,.$$

Cette définition s'étend à des états quantiques normalisés $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ dans l'espace de Fock, i.e pour toute suite $(\varepsilon_n)_{n\geqslant 0}$ vérifiant $\lim_{n\to+\infty}\varepsilon_n = 0$, il existe une sous-suite $(\varepsilon_{n_k})_{k\geqslant 0}$ telle que $\lim_{k\to+\infty}\varepsilon_{n_k} = 0$ et

$$\forall \xi \in L^2(\mathbb{R}^d),\ \lim_{k\to+\infty} \mathrm{Tr}\,[\mathcal{W}(\sqrt{2}\pi\xi)\varrho_{\varepsilon_{n_k}}] = \int_{L^2(\mathbb{R}^d)} e^{2i\pi\,\mathrm{Re}\,\langle \xi, z\rangle}\, d\mu(z).$$

Un résultat montré dans [9] affirme que l'ensemble $\mathcal{M}(\varrho_N,\, N \in \mathbb{N}^*)$ est toujours non-vide (voir plus de détails dans la Section 2.2). Donc, quitte à extraire une sous-suite, on peut toujours supposer que la famille d'états $(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|)_{N\in\mathbb{N}}$ admet une unique mesure de Wigner μ. Les états quantiques évolués par la dynamique

$$(\varrho_N(t) = |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}|)_{N\in\mathbb{N}}$$

admettent une famille de mesures de Wigner $\mathcal{M}(\varrho_N(t),\, N \in \mathbb{N}^*)$ pour chaque instant $t \in \mathbb{R}$.

> **L'approximation de champ moyen est effective** si $\mathcal{M}(\varrho_N(t),\, N \in \mathbb{N}^*) = \{\mu_t\}$ où μ_t est **la mesure image de μ par le flot de l'équation (Hartree Générale) classique**.

Ainsi, la mesure de Wigner est transportée dans l'espace des phases par un flot non-linéaire. De plus, des estimations a priori au niveau quantique donne des informations sur la localisation de la mesure dans l'espace de phase. La méthode proposée dans [12] consiste à résoudre une équation de transport satisfaite par la famille $(\mu_t)_{t\in\mathbb{R}}$, dans un sens faible (précisée dans (1.1.15))

$$\partial_t \mu_t + \nabla^T(v_t(z)\mu_t) = 0, \qquad \textbf{(Transport de mesures)}$$

avec $v_t(z)$ un champ de vitesse correspondant à l'équation (**Hartree**) dans sa représentation d'interaction:

$$\begin{cases} \partial_t z = v_t(z) = -ie^{itA}\big[W * |e^{-itA}z|^2(e^{-itA}z)\big] \\ z_{t=0} = z_0. \end{cases} \tag{1.1.14}$$

Détaillons à présent la méthode permettant d'obtenir l'équation (**Transport de mesures**). Commençons par la dérivation par rapport au temps t de la quantité

$$\mathcal{I}_N(t) = \langle e^{-itH_N^0}e^{itH_N}\Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)e^{-itH_N^0}e^{itH_N}\Psi^{(N)}\rangle.$$

Remarquons que l'on ne s'intéresse pas directement à la solution de l'équation (**Schrödinger à N corps**) mais plutôt à celle dans sa représentation d'interaction, i.e.

$$\tilde{\Psi}_t^{(N)} := e^{-itH_N^0}e^{itH_N}\Psi^{(N)}\,,$$

avec $(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|)_{N\in\mathbb{N}}$ la famille d'états normaux initiaux. La régularité de la fonction $t \mapsto \mathcal{I}_N(t)$ s'obtient pour $\xi \in \mathcal{D}$ où $\mathcal{D}$ désigne un domaine dense dans $L^2(\mathbb{R}^d)$. Ainsi on peut établir une équation intégrale sous la forme suivante pour tout $s, t \in \mathbb{R}$

$$\mathcal{I}_N(t) = \mathcal{I}_N(s) + i\int_s^t \langle\tilde{\Psi}_u^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\big[\sum_{j=1}^4 \frac{1}{N^{j-1}}\mathcal{O}_j(\xi,u)^{Wick}\big]\tilde{\Psi}_u^{(N)}\rangle\, du, \qquad \textbf{(Equation Intégrale)}$$

où $\mathcal{O}_j(\xi,u)$ désigne des polynômes de Wick. Sous une hypothèse de compacité (concernant les états normalisés $\Psi^{(N)}$ dans [11] sous la condition (**PI**) ou en considérant une perturbation W relativement compacte du Laplacien au sens des formes dans [12]), le passage à la limite aboutit à l'équation de Liouville suivante

$$\tilde{\mu}_t(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}) = \tilde{\mu}_s(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}) + i\int_s^t \tilde{\mu}_u(\{q_u(z), e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}\})\, du,\ s<t, \qquad \textbf{(Liouville)}$$

avec $q_u(z) = \frac{1}{2}\langle e^{-iuA}z^{\otimes 2}, W_{1,2}\, e^{-iuA}z^{\otimes 2}\rangle$ qui désigne la partie interaction de (**Energie de Hartree**). La convergence de (**Equation Intégrale**) n'est valable qu'après l'extraction d'une sous-suite pour assurer l'existence de la famille de mesures de Wigner $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ associées à la famille $(|\tilde{\Psi}_t^{(N)}\rangle\langle\tilde{\Psi}_t^{(N)}|)$. Notons $\mathfrak{p}$ une projection orthogonale de rang fini et $\mathfrak{L}(\mathfrak{p}L^2(\mathbb{R}^d))$ la mesure de Lebesgue associée au sous espace de dimension finie $\mathfrak{p}L^2(\mathbb{R}^d)$. On conclut par intégrer l'expression (**Liouville**) par $\mathcal{F}[g](\xi)\mathfrak{L}(\mathfrak{p}L^2(\mathbb{R}^d))$ où $\mathcal{F}[g](\xi)$ désigne la transformée de Fourier d'une fonction g régulière évaluée en $\xi \in \mathcal{D}$. On obtient par densité l'expression suivante qui est l'équation de transport au sens faible, i.e

$$\int_{\mathbb{R}\times\mathcal{D}} \partial_t f(t,z) + i\{q_t(z), f(t,z)\}\, d\tilde{\mu}_t\, dt = 0, \tag{1.1.15}$$

avec f une fonction lisse à support compact sur $\mathbb{R}\times\mathcal{D}$ qui ne dépend que d'un nombre fini de variables. On peut à présent énoncer le résultat principal de [12] sous les hypothèses:

1. $A = -\Delta$, $W_{i,j} = W(x_i - x_j)$, W est une fonction paire mesurable,

2. $W(1-\Delta)^{-\frac{1}{2}}$ est un opérateur borné de $L^2(\mathbb{R}^d)$ dans $L^2(\mathbb{R}^d)$,
3. $(1-\Delta)^{-\frac{1}{2}}W(1-\Delta)^{-\frac{1}{2}}$ est un opérateur compact.

Theorem 1.1.5. *Soit $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ une famille d'états quantiques normalisés dans l'espace de Fock, admettant une unique mesure de Wigner μ_0 telle que*

$$\mathrm{Tr}\left[(\mathbf{N}+H_\varepsilon^0)^\delta \varrho_\varepsilon\right] \leqslant C_\delta < \infty, \tag{1.1.16}$$

uniformément par rapport à $\varepsilon \in (0,\bar{\varepsilon})$ et pour un $\delta > 0$. Alors, pour tout temps $t \in \mathbb{R}$, la famille $(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar{\varepsilon})}$ admet une unique mesure de Wigner μ_t. C'est également une mesure de probabilité sur l'espace d'énergie $H^1(\mathbb{R}^d)$ et on a l'égalité $\mu_t = \varphi(t,0)*\mu_0$. Ainsi la mesure μ_t est la mesure image de μ_0 par le flot de l'équation (**Hartree**)*, globalement bien définie sur $H^1(\mathbb{R}^d)$.*

En notant l'espace $L^p(\mathbb{R}^d)+L_0^\infty(\mathbb{R}^d)$ des fonctions f mesurables telles qu'il existe une suite $(f_n)_{n\in\mathbb{N}} \in L^p(\mathbb{R}^d)$ avec $p > 2$ vérifiant $\lim_{n\to+\infty}\|f-f_n\|_{L^\infty(\mathbb{R}^d)} = 0$. Ainsi, l'approximation de champ moyen est effective pour des potentiels de type $L^2(\mathbb{R})+L_0^\infty(\mathbb{R})$, $L^p(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$ pour $p > 2$ et $W = \frac{\lambda}{|x|}$, $\lambda \in \mathbb{R}$ en dimension 3 ou plus généralement en dimension $d \geqslant 3$, les potentiels $W \in L_{\mathrm{w}}^d(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)$ dans un cadre non confinant $A = -\Delta$. Cette approche permet ainsi d'obtenir une grande diversité d'états quantiques sous la seule hypothèse (1.1.16). Cette hypothèse est naturelle dans le sens où elle prévient la perte de masse et d'énergie à l'infini. Cette méthode implique également la convergence des matrices à densité réduite (1.1.9). On retrouve aussi la convergence des états cohérents

$$\varrho_\varepsilon(\varphi) = |\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\varphi)\Omega\rangle\langle\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\varphi)\Omega|,\ \varphi\in L^2(\mathbb{R}^d), \Omega = (1,0,...)\in\Gamma_s(L^2(\mathbb{R}^d)), \tag{1.1.17}$$

via l'équation suivante pour des observables de Wick (définis dans la Section 2)

$$\lim_{\varepsilon\to 0}\langle e^{-it\varepsilon^{-1}H_\varepsilon}\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\varphi)\Omega, b^{Wick}e^{-it\varepsilon^{-1}H_\varepsilon}\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\varphi)\Omega\rangle = b(\varphi_t), \tag{1.1.18}$$

φ_t étant solution d' (**Hartree**) avec pour donnée initiale f.

Je terminerai cet historique par un phénomène qui a connu un grand intérêt depuis une vingtaine d'années, l'apparition de "condensats", dits de Bose-Einstein. Ces derniers apparaissent expérimentalement pour des atomes ultra-froids en 1995 et ont été prévus par Einstein dans les années 20. Ces condensats sont formés par des particules bosoniques qui occupent dans une proportion "macroscopique" le même état quantique. Mathématiquement, Gross dans [60, 59] et Pitaevskii dans [96] ont décrit l'évolution d'un condensat de Bose-Einstein par l'équation susnommée

$$i\partial_t z_t = -\Delta z_t + 8\pi a|z_t|^2 z_t, \tag{Gross-Pitaevskii}$$

avec a désignant la longueur de diffusion (voir par exemple [115] pour une définition précise). Le modèle de gaz de bosons dilués approche quand le nombre de particules N est grand l'équation non linéaire (**Gross-Pitaevskii**). Le hamiltonien microscopique décrivant ce système de bosons est donné par

$$H_N = \sum_{i=1}^N A_i + \frac{1}{N}\sum_{1\leqslant i<j\leqslant N} W_N(x_i - x_j), \tag{1.1.19}$$

avec $W_N(x) = N^{\beta d} W(N^\beta x), \beta \in [0,1]$ et W est une fonction mesurable régulière, à symétrie sphérique, décroissant assez vite à l'infini. Ce changement d'échelle dans l'interaction fut introduit par Lieb, Yngvason et Seiringer dans [84]. Ainsi, différents régimes apparaissent dans la limite $N \to +\infty$. Le cas $\beta = 0$ correspond à la limite champ moyen alors que $\beta = 1$ donne la convergence suivante

$$\lim_{N\to+\infty} \operatorname{Tr}\left[\left|\gamma_{N,k}^t - |\Psi_{\mathrm{GP}}(t)\rangle\langle\Psi_{\mathrm{GP}}(t)|^{\otimes k}\right|\right] = 0, \tag{1.1.20}$$

avec Ψ_{GP} satisfaisant l'équation (**Gross-Pitaevskii**) et $\gamma_{N,k}^t$ désignant l'évolution des matrices à densité réduite à k particules. Donc, dans le cas $\beta = 1$, l'équation limite est celle de (**Gross-Pitaevskii**) et cela a été prouvé dans des nombreux travaux (voir par exemple [42, 43, 20]). Dans le cas $0 < \beta < 1$, la constante de couplage dans l'équation limite n'est plus $8\pi a$ mais plutôt $b_0 = \int_{\mathbb{R}^d} W(x)dx$, et une convergence analogue à (1.1.20) est prouvée dans plusieurs travaux ([1] en dimension un, [40] en dimension 3). Les méthodes utilisées pour traiter cette convergence avec une interaction "moins forte" sont souvent basées sur les hiérarchies **BBGKY** pour traiter des états initiaux factorisés ou cohérents. Dans la plupart des cas étudiés dans la littérature citée, l'opérateur A est de la forme $-\Delta + U$ pour décrire des particules non-relativistes en l'absence de champ magnétique. Le modèle de gaz de bosons dilués a également fait l'objet de travaux d'un point de vue variationnel, et en particulier dans le cadre "champ moyen": $\beta = 0$, comme on le verra dans la suite.

1.1.2 Approximation de champ moyen vers l'état fondamental

Dans cette sous-section, on considère que l'Hamiltonien H_N est semi-borné inférieurement. L'état fondamental du système de N particules bosoniques E_N est donné par l'infimum de la fonctionnelle

$$\mathcal{E}_Q(\Psi^{(N)}) = \langle \Psi^{(N)}, H_N \Psi^{(N)} \rangle_{L^2(\mathbb{R}^{dN})}$$

sous la condition de masse $\|\Psi^{(N)}\|_{L^2(\mathbb{R}^{dN})} = 1$, i.e

$$E_N = \inf_{\Psi^{(N)} \in Q(H_N),\, \|\Psi^{(N)}\|=1} \mathcal{E}_Q(\Psi^{(N)}), \tag{1.1.21}$$

où $Q(H_N)$ désigne le domaine forme de l'opérateur H_N. Ainsi l'état fondamental correspond au minimum du spectre de l'opérateur H_N. Notons Ψ un minimiseur (approché) de l'énergie classique de l'équation (**Hartree Générale**) sous la contrainte $||\Psi|| = 1$. Dans ce cadre variationnel, l'approximation de champ moyen se traduit quand le nombre de particules N est grand par

$$\frac{E_N}{N} \approx h(\Psi, \overline{\Psi})\,. \tag{1.1.22}$$

Quel sens mathématique donné à cette approximation? Comme on l'a vu précédemment, l'énergie $\mathcal{E}_Q(\Psi^{(N)})$ est de l'ordre de N. Ainsi le champ moyen nécessite un changement d'échelle en $\frac{1}{N}$ et en supposant que l'énergie classique de (**Energie de Hartree**) est bornée inférieurement, on obtient la formulation mathématique équivalente à (1.1.22):

Theorem 1.1.6.

$$\lim_{N\to+\infty} \frac{E(N)}{N} = \lim_{N\to+\infty} \frac{1}{N} \inf_{\|\Psi^{(N)}\|=1,\, \Psi^{(N)}\in Q(H_N)} \mathcal{E}_Q(\Psi^{(N)}) = \inf_{z\in Q(h(z,\bar z)),\|z\|_{\mathcal{Z}_0}=1} h(z,\bar z) > -\infty. \tag{1.1.23}$$

Historique des résultats sur l'approche variationnelle de la limite de champ moyen.

Le théorème 1.1.6 a donné lieu a de nombreux travaux pour de nombreux modèles différents: des bosons avec ou sans influence de champ magnétique, ou encore des considérations plus générales pour des modèles de gaz de bosons dilués (voir discussion dans le précédent historique), en prenant en compte soit des particules non-relativistes, soit des particules semi-relativistes. Les premiers résultats établissaient une borne supérieure pour la quantité $\frac{E(N)}{N}$, ce qui fut effectué par Dyson pour un potentiel d'interaction de type "sphère dure" en 1957 dans [37]. La borne inférieure a été obtenue par Lieb et Yngvason dans [85, 84] pour un potentiel d'interaction plus général (décroissant relativement vite à l'infini) avec $A = -\Delta$ et un gaz de bosons dans une boite de taille L. La convergence (1.1.6) a été prouvée dans le cadre d'atomes satisfaisant la statistique de Bose-Einstein dans les travaux [21, 108, 15, 56, 77, 67]. Le Hamiltonien considéré dans ce cadre est le suivant

$$H_N = \sum_{i}^{N} \big[-\Delta_{x_i} - \frac{1}{t|x_i|}\big] + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} \frac{1}{|x_i - x_j|}, \tag{1.1.24}$$

avec t une donnée inhérente au système. Le modèle de Lieb-Liniger a également fait l'objet d'une étude dans les travaux [83, 80, 105]. Il correspond à l'étude de gaz de bosons en dimension un par l'intermédiaire du Hamiltonien H_N

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial_{x_j^2}} + \sum_{1\leqslant i<j\leqslant N} \delta(x_i - x_j), \tag{1.1.25}$$

où δ est la distribution de Dirac. La plupart des résultats concernant l'approximation du Théorème (1.1.6) sont liés à la théorie perturbative de Bogoliubov introduite en 1947 dans [24].

Limite thermodynamique:

Dans l'étude de gaz de bosons dilués dans une boite Λ de taille L en dimension 3, l'Hamiltonien est donné par l'expression (1.1.19) avec $\beta = 0$ et un potentiel W très régulier, positif décroissant plus vite que $\frac{1}{r^3}$ à l'infini. L'état fondamental $E(N, L)$ dans ce cadre dans la limite, dite thermodynamique quand N et L tendent vers l'infini avec une densité de particules fixée donnée par $\varrho = \frac{N}{|\Lambda|}$. L'énergie par particules est dénotée

$$e_0(\varrho) := \lim_{L\to+\infty} \frac{E(\varrho L^3, L)}{\varrho L^3}.$$

La limite pour une densité faible est donnée par l'égalité

$$\lim_{\varrho a^3\to 0} \frac{e_0(\varrho)}{4\pi\varrho a} = 1,$$

avec a la longueur de diffusion associée au potentiel W (voir [115] pour une définition détaillée). Pour des gaz de bosons dilués, l'idée est de considérer l'énergie associée à chaque paire de particules comme étant "asymptotiquement" indépendante dans l'esprit de l'approximation suivante

$$E(N, L) \approx \frac{1}{2} N(N-1)E(2, L) \approx \frac{1}{2} N^2 8\pi a \frac{1}{L^3} = N 4\pi a\varrho.$$

La convergence dans ce cadre a été étudiée, en dimension 3, par Lieb, Seiringer, Solovej et Yngvason dans [87, 85, 82] et dans [84] pour des bosons piégés en dimension 2 (voir aussi [86]). Les ordres supérieurs pour cette convergence ont fait l'objet des travaux [81, 71, 55, 41] ou [57] pour des bosons confinés, et sont toujours basés sur la méthode développée par Bogoliubov. Dans [76] les auteurs ont étendus les résultats de [57] pour des gaz de bosons dilués confinés en dimension $2, 3$ dans un cadre très général incluant également le cadre d'atomes bosoniques (voir l'hamiltonien (1.1.24)) par l'intermédiaire d'arguments de localisation dans l'espace de Fock et l'existence du Hamiltonien de Bogoliubov $\mathcal{H}$ via l'approximation suivante quand $N \to +\infty$

$$H_N \approx N \inf_{z \in Q(h), \|z\|_{\mathcal{Z}_0}=1} h(z, \bar{z}) + \mathcal{H} + o(1).$$

L'approximation de champ moyen (de Hartree) donne le premier ordre et la théorie perturbative de Bogoliubov prévoit le premier et les termes suivants en étudiant $\mathcal{H}$.

Limite de Gross-Pitaevskii:

L'approximation vers l'état fondamental a également été motivée par l'étude des condensats de Bose-Einstein, et en particulier par un autre type de limite précédemment évoqué, la limite de Gross-Pitaevskii. Le modèle de N bosons correspondant a été introduit dans (1.1.19). Par simplicité, exposons le cadre de la dimension 3, où le Hamiltonien est

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} N^3 W(N(x_i - x_j)),$$

avec W un potentiel régulier. La limite de Gross-Pitaevskii est un cas particulier du cadre des gaz de bosons dilués. En effet, cela revient à considérer la limite $N \to +\infty$ avec les rapport constants $\frac{Na}{L}$ et $g := \frac{4\pi N a}{L} \approx \frac{e_0}{e_V}$ avec a désignant la longueur de diffusion et L la taille de la boite $L = |\Lambda|^{\frac{1}{3}}$ et e_V désignant le trou spectral de l'opérateur $-\Delta + V$. Nous renvoyons à [115] ou [82] pour des présentations détaillées sur les différents résultats de convergence des différentes limites évoquées. Le théorème 1.1.6 est ainsi vrai avec $h(z, \bar{z})$ désignant l'énergie classique associée l'équation de (**Gross-Pitaevskii**) et fut démontré en particulier par Lieb, Yngvason, Solovej et Seiringer dans [87].

Approche géométrique:

L'approche récente de Lewin, Phan et Rougerie est basée sur une méthode géométrique introduite dans [74] pour démontrer la convergence du Théorème (1.1.6). l'Hamiltonien à N corps est donné par l'expression

$$H_N^V = \sum_{i=1}^{N} (-\Delta_{x_i} + V(x_i)) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} W(x_i - x_j) \tag{1.1.26}$$

Ainsi, l'idée est d'utiliser le théorème (**HVZ**) pour décrire le lien entre le spectre essentiel de l'Hamiltonien H_N^V pour N particules et l'état fondamental du Hamiltonien à $N - k$ particules $E^V(N-k)$ et celui à k particules $E^0(k)$. Moralement, on peut atteindre le bas du spectre essentiel en envoyant k particules du système à l'infini. Ainsi, l'énergie totale du système est la somme de l'énergie $E^V(N-k)$ et l'énergie des particules placées à l'infini $E^0(k)$:

$$\inf \sigma_{\text{ess}}(H^V(N)) = \inf\{E^V(N-k) + E^0(k),\ k = 1, \cdots, N\}. \tag{\textbf{HVZ}}$$

La convergence du Théorème (1.1.6) peut-être traitée dans le cadre abstrait donné par l'Hamiltonien H_N (1.1.2) en utilisant un argument de localisation dans l'espace de Fock combiné avec les théorèmes de de Finetti quantiques. Pour des bosons confinés, l'approximation de champ moyen peut être traitée sans utiliser le théorème (**HVZ**). Décrivons brièvement cette approche.
Rappelons la définition (1.1.8) des matrices à densité réduite $\gamma_{k,N}$:

$$\gamma_{k,N} := \text{ Tr }_{[k+1,N]}|\Psi^{(N)}\rangle\langle\Psi^{(N)}|,\ 0 \leqslant k \leqslant N. \tag{1.1.27}$$

L'énergie par particules peut ainsi s'écrire en termes des matrices à densité réduite à un et deux corps,

$$\frac{\langle\Psi^{(N)}, H_N\Psi^{(N)}\rangle}{N} = \text{ Tr }[A\gamma_{1,N}] + \frac{1}{2}\text{ Tr }_{\otimes^2\mathcal{Z}_0}[W_{1,2}\gamma_{2,N}] =: \frac{1}{2}\text{ Tr }_{\otimes^2\mathcal{Z}_0}[H_2\gamma_{2,N}]. \tag{1.1.28}$$

Ainsi, on peut reformuler l'approximation vers l'état fondamental du système par

$$\frac{E(N)}{N} = \frac{1}{2}\inf\{\text{ Tr }_{\otimes^2\mathcal{Z}_0}[H_2\gamma_{2,N}],\ \gamma_{2,N} \in \mathcal{P}_N^{(2)}\} \tag{1.1.29}$$

où $\mathcal{P}_N^{(2)} := \{\gamma_2 \in \mathcal{L}^1(\otimes^2\mathcal{Z}_0),\ \exists\Psi^{(N)} \in \otimes^N\mathcal{Z}_0,\ \|\Psi^{(N)}\| = 1,\ \gamma_2 = \gamma_{2,N}\}$. Ainsi, il suffit de décrire l'espace $\mathcal{P}_N^{(k)}$ dans la limite $N \to +\infty$ et en particulier l'espace $\mathcal{P}^{(2)}$ défini par l'égalité suivante $\mathcal{P}^{(2)} := \bigcap_{N\geqslant 1}\mathcal{P}_N^{(2)}$. Le théorème de de Finetti quantique permet de répondre à cette question, i.e.

Theorem 1.1.7. *Soit $\mathcal{Z}_0$ un espace de Hilbert séparable et $(\gamma_k)_{k\in\mathbb{N}} : \bigvee^k\mathcal{Z}_0 \to \bigvee^k\mathcal{Z}_0$ une suite d'opérateurs autoadjoints positifs. On suppose que pour tout $k, n \in \mathbb{N}$*

$$\text{Tr }_{[k+1,k+n]}[\gamma_{k+n}] = \gamma_k.$$

On suppose de plus que $\gamma_0 = 1$. Alors il existe une unique mesure de probabilité μ sur la sphère unité $\mathcal{S}_{\mathcal{Z}_0}$, telle que

$$\gamma_k = \int_{\mathcal{S}_{\mathcal{Z}_0}} |z^{\otimes k}\rangle\langle z^{\otimes k}|\, d\mu(z),\ \forall k \geqslant 0. \tag{1.1.30}$$

De plus si $\text{ Tr }[A\gamma_1] < +\infty$ pour un opérateur autoadjoint $A \geqslant 0$ sur $\mathcal{Z}_0$, alors μ vérifie l'égalité

$$\mu(\mathcal{S}_{\mathcal{Z}_0}\backslash Q(A)) = 0. \tag{1.1.31}$$

Ce théorème est en lien avec un résultat de Stormer et Hudson-Moody dans [38, 65] qui généralise un théorème classique de de Finetti (aussi appelé Hewitt-Sauvage). Il existe une version faible de ce résultat présentée dans [77]. Ce dernier est un cas particulier des convergences pour les mesures de Wigner introduit par Ammari et Nier dans [11, 12] et qui feront l'objet d'un rappel dans la Section 2.2.

L'égalité (1.1.30) permet d'écrire les éléments γ_k de $\mathcal{P}^{(k)}$ au moyen d'une mesure de probabilité μ sur la sphère $\mathcal{S}_{\mathcal{Z}_0}$. Ainsi la convergence (1.1.6) se prouve en remarquant (formellement) que

$$\frac{1}{2}\operatorname{Tr}_{\otimes^2\mathcal{Z}_0}[H_2\gamma_{2,N}] \to \frac{1}{2}\int_{\mathcal{S}_{\mathcal{Z}_0}} \langle z^{\otimes 2}, H_2 z^{\otimes 2}\rangle \, d\mu(z) = \int_{\mathcal{S}_{\mathcal{Z}_0}} h(z,\bar{z})\, d\mu(z) \geqslant \inf_{z\in Q(h),\, \|z\|_{\mathcal{Z}_0}=1} h(z,\bar{z}), \tag{1.1.32}$$

car μ est une mesure de probabilité sur la sphère $\mathcal{S}_{\mathcal{Z}_0}$. Dans un cadre très abstrait, avec ou sans champ magnétique, sous l'hypothèse que l'Hamiltonien libre à une particule ait une résolvante compacte, l'approximation (1.1.6) est vérifiée pour des potentiels satisfaisant des conditions de décroissance "naturelles" (potentiel à deux corps relativement borné par rapport au Hamiltonien libre à une particule). Pour un système non confiné, la version faible du théorème de de Finetti ne suffit pas. Ainsi, dans le cas invariant par translation ($A = -\Delta$ et $W_{i,j} = W(x_i - x_j)$), le manque de compacité se gère au moyen d'inégalité de liaison "binding inequalities" (conséquence du théorème (**HVZ**)), i.e. $E^V(N)$ une valeur propre isolée si et seulement si

$$E^V(N) < E^V(N-k) + E^0(k),\ \forall k = 1,\cdots,N.$$

Au lieu de considérer une inégalité de liaison au niveau quantique, Lewin, Nam et Rougerie supposent uniquement cette inégalité vraie pour l'état fondamental de l'énergie classique (**Energie de Hartree**) et ainsi le résultat [77, Théorème 4.5] valide l'approximation de champ moyen sous des hypothèses de décroissance des potentiels V, W. Nous terminerons cette section par mentionner que les mêmes auteurs ont adapté cette méthode également pour un système de bosons confinés dans le cadre de la limite de **Gross-Pitaevskii** (voir [75]).

1.2 Problématique et résultats

Nous souhaitons établir une méthode générale qui inclut les résultats précédents et qui les complète. L'idée est de rendre effectif l'approximation dynamique de champ moyen via l'approche des mesures de Wigner pour un Hamiltonien H_N abstrait introduit dans (1.1.2). Le résultat d'Ammari et Nier 1.1.5 peut également être complété dans un cadre d'interaction avec un nombre fini de corps sous une condition de compacité pour une famille générale d'états normaux dans l'espace de Fock. Dans le cadre d'une interaction à deux corps, en travaillant 'localement' dans l'espace de Fock, l'approximation de champ moyen sera validée pour des états quantiques d'énergies finies, i.e

$$\exists C > 0,\ \langle \Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle \leqslant CN, \tag{1.2.1}$$

pour des potentiels plus singuliers. L'idée est de valider le théorème 1.1.5 pour des potentiels W singuliers, sous critiques, i.e pour lesquels l'équation d'(**Hartree**) est globalement bien posée. L'existence et l'unicité de l'équation de Hartree a été traitée dans le cas non relativiste sans champ magnétique par Ginibre et Vélo dans [54] pour des potentiels très singuliers (décroissance à l'infini et singularité en 0), ou dans un cadre avec champ magnétique par [91], ou encore abstrait pour un hamiltonien libre A borné inférieurement par [28]. Cependant certains potentiels ne garantissent pas l'existence d'une dynamique quantique, ni même le caractère autoadjoint de H_N. Il apparait ainsi que l'on souhaiterait valider l'approximation de champ moyen dans le cadre d'un espace de Hilbert séparable $\mathcal{Z}_0$ pour des interactions relativement bornées par rapport à l'hamiltonien libre A (en utilisant les théorèmes de Kato-Reillich ou KLMN [100]). Nous verrons par la suite que le caractère infinitesimal fournit une condition suffisante (au moins dans le cadre de particules confinées) de validation du champ moyen. Atteindre des potentiels peu réguliers ou coulombiens de type $W = \frac{1}{|.|^\alpha}$, $1 < \alpha < 2$ pour des états quantiques normaux très généraux satisfaisant (1.2.1) n'a pas été traité dans la littérature. La compréhension de la méthode basée sur l'étude d'une équation de transport sur les mesures de Wigner permet de valider l'approximation de champ moyen dans un cadre abstrait, pour des bosons piégés (cadre confinant), i.e

$$(A+i)^{-1} \in \mathcal{L}^\infty(\mathcal{Z}_0), \tag{1.2.2}$$

avec ou sans influence de champ magnétique.
Cependant les hypothèses des théorèmes de l'appendice C dans [12] ne permettent pas d'atteindre des potentiels singuliers, et cela s'explique en particulier par le contrôle fort imposé sur le champ de vitesse

$$v_t(z) = -ie^{itA}\big[(W * |e^{-itA}z|^2)e^{-itA}z\big],$$

donné par

$$\forall T > 0, \int_0^T \|v_t(z)\|_{L^2(Q(A),\, d\mu_t(z))}\, dt < \infty.$$

Ainsi la résolution du (**Transport de mesures**) en dimension infinie pour des potentiels plus singuliers nécessite l'affaiblissement des hypothèses de contrôle du champ de vitesse $v_t(z)$ en utilisant les travaux [5, 89, 12]. L'hypothèse la plus naturelle dans ce sens que l'on retrouve également dans le travail de [69] est l'estimation suivante

$$\exists C > 0,\, \forall z \in Q(A),\; \|v_t(z)\|_{\mathcal{Z}_0} \leqslant C\|z\|^2_{Q(A)}\|z\|_{\mathcal{Z}_0}. \tag{1.2.3}$$

Combinée cette estimation avec les informations a priori obtenues via la définition des mesures de Wigner permet le contrôle du champ de vitesse suivant

$$\forall T > 0, \int_0^T \|v_t(z)\|_{L^1(\mathcal{Z}_0,\, d\mu_t(z))}\, dt < \infty. \tag{1.2.4}$$

Cette dernière équation constituera la condition suffisante d'existence de solutions 'faibles' à l'équation (**Transport de mesures**), en ajoutant une hypothèse de continuité faible sur les mesures de Wigner. Ce type de résolution d'une équation de transport sur des mesures a été étudié dans un cadre plus général en dimension finie pour le contrôle (1.2.4) dans [89] mais n'a jamais été traité en dimension infinie.
Comme dans le cadre des hiérarchies **BBGKY**, le problème de la perte de compacité liée à la dimension infinie constitue l'obstacle majeur pour l'obtention de la limite de champ moyen. Dans le cadre de la stratégie basée sur les mesures de Wigner, la difficulté vient lors de l'établissement de
(**Transport de mesures**). La perte de compacité peut être compensée d'au moins deux façons différentes.
La première consiste à considérer des états quantiques normaux dans l'espace de Fock satisfaisant la condition (**PI**), introduite dans [11] donnée par l'égalité suivante

$$\lim_{\varepsilon \to 0} \operatorname{Tr}\,[\varrho_\varepsilon \mathbf{N}^k] = \int_{\mathcal{Z}_0} \|z\|^{2k}_{\mathcal{Z}_0} d\mu(z), \;\; \forall k \in \mathbb{N}\,, \tag{1.2.5}$$

où μ désigne la mesure de Wigner associée à la famille d'états normaux $(\varrho_\varepsilon)_{\varepsilon\in(0,1)}$. Pour des états normaux 'localisés' dans l'espace de Fock, i.e $\|\Psi^{(N)}\|_{\bigvee^N \mathcal{Z}_0} = 1$ l'égalité se traduit par

$$\int_{\mathcal{Z}_0} \|z\|^2_{\mathcal{Z}_0}\, d\mu(z) = 1.$$

Cette condition (**PI**) est vérifiée dans un cadre abstrait pour un hamiltonien libre généré par un opérateur autoadjoint positif A à résolvante compacte et pour des états satisfaisant (1.2.1). Ainsi, nous montrerons que la limite de champ moyen est validée dans le cadre abstrait du hamiltonien (1.1.2) pour un opérateur A à résolvante compacte en supposant le caractère infinitesimallement borné du potentiel $W_{1,2}$ à deux corps par rapport à l'opérateur $A_1 + A_2 := A\otimes\ \mathrm{Id}\ +\ \mathrm{Id}\ \otimes A$. Il est possible de relaxer cette dernière hypothèse en supposant que le noyau de l'interaction $q(z^{\otimes 2}, z^{\otimes 2}) := \frac{1}{2}\langle z^{\otimes 2}, W_{1,2} z^{\otimes 2}\rangle$ soit relativement borné par rapport à $A_1 + A_2$ mais il faut néanmoins supposer le caractère infinitésimal du noyau de la dérivée de $q(z^{\otimes 2}, z^{\otimes 2})$ (voir (**D1**) Section 5 pour une présentation détaillée).
La deuxième consiste à considérer l'approximation de champ moyen dans le cadre non relativiste $A = -\Delta$, $\mathcal{Z}_0 = L^2(\mathbb{R}^d)$ et pour des potentiels à deux corps générés par un potentiel W qui lui-même sera une perturbation relativement compacte par rapport à $-\Delta$ au sens des formes, i.e

$$(1-\Delta)^{-\frac{1}{2}} W (1-\Delta)^{-\frac{1}{2}} \in \mathcal{L}^\infty(\mathcal{Z}_0). \tag{1.2.6}$$

Cette dernière condition est vérifiée pour des potentiels singuliers (elle est vraie également pour $W = \frac{1}{|.|^\alpha}$, $0 < \alpha < 2$, $d = 3$) et nous montrerons que l'approximation de champ moyen devient effective sous cette condition (qui induit l'existence de la dynamique quantique par le théorème KLMN) et sous la condition (1.2.1). Cela complétera le théorème 1.1.5 mais on perdra le caractère abstrait du cadre confinant. Une condition plus forte sera également présentée dans la Section 5, hypothèse (**D2**) (le noyau de la dérivée de l'interaction q est une perturbation relativement compacte).

1.3 Plan de la thèse

Ce manuscrit s'organise de la façon suivante. Nous commençerons dans le Chapitre 2 par introduire le vocabulaire de la seconde quantification et les principaux outils et définitions (espace de Fock, opérateurs de création-annihilation, nombre, opérateur de champ, opérateur de Weyl). La quantification de Wick ou ordre de Wick pour des symboles polynomiaux bornés (classe $\mathcal{P}_{p,q}$) fera l'objet d'une introduction approfondie en dimension finie et infinie. Les principales caractéristiques associées à des observables de Wick (formules de commutations, produit, conjugaison) seront également étudiées dans la Section 2.1.3. On rappellera également les quantifications de Weyl et Anti-Wick dans la sous-section 2.1.1 ainsi que les correspondances entre les opérateurs principaux de l'espace de Fock et les opérateurs de Toeplitz. La quantification de l'énergie du champ moyen (**Energie de Hartree**) nécessite la construction d'une classe de symboles plus large, notée $\mathcal{Q}_{p,q}(A)$ dépendant de l'opérateur positif A. Dans la sous-section 2.1.4, on abordera les propriétés inhérentes à cette classe de symboles polynomiaux à noyau non borné. Dans la Section 2.2, on rappellera la définition des mesures semiclassiques en dimension finie puis en dimension infinie. Le défaut de compacité sera également l'objet d'une discussion au niveau des états quantiques et des observables, via la condition (**PI**). On finira la Section 2 par l'étude des relations entre les classes de symboles de Wick $\mathcal{P}_{p,q}$ et $\mathcal{Q}_{p,q}(A)$ et les mesures de Wigner au

travers de plusieurs estimations a priori.
Dans le Chapitre 3, on commencera par une brève introduction sur les équations de Liouville en dimension finie. Ensuite, on abordera la question des équations de transport satisfaites par des mesures de probabilité. Dans la Section 3.1, nous proposerons un résultat d'unicité sur les mesures de probabilité boréliennes d'une équation de Liouville (ou continuité ou transport) via la méthode des caractéristiques en combinant les arguments présents dans les travaux [5, 12, 89]. Dans la section 3.2, on donnera plusieurs exemples d'EDP non-linéaires pour lesquelles on peut appliquer notre résultat d'unicité.
Le Chapitre 4 apporte le premier résultat de champ moyen prouvé dans ce manuscrit. Ce résultat a été obtenu en collaboration avec Boris Pawilowski. Il concerne un système de N bosons identiques décrit par l'Hamiltonien suivant

$$H_\varepsilon^{(n)} = H_\varepsilon^{0,(n)} + \sum_{\ell=2}^{r} \varepsilon^\ell \frac{n!}{(n-\ell)!} \mathcal{S}_n(\tilde{Q}_\ell \otimes \mathrm{Id}_{\bigvee^{n-\ell}\mathcal{Z}})\mathcal{S}_n\,, \quad n \geqslant 2r\,, \tag{1.3.1}$$

avec $\varepsilon \to 0\,, n\varepsilon \to 1$. Ici les $\tilde{Q}_\ell$'s sont des opérateurs symétriques bornés sur $\bigvee^\ell \mathcal{Z}$ et

$$H_\varepsilon^{0,(n)} = \varepsilon \sum_{i=1}^{n} \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \underbrace{A}_{i} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}\,, \tag{1.3.2}$$

avec A un opérateur autoadjoint. Le modèle proposé ici fut étudié par Ammari et Nier dans [12]. Le cas d'une interaction bornée à plusieurs corps a aussi été traité dans [10] sous la condition (**PI**) vérifiée par les états normaux initiaux. Notre travail dans ce modèle permet la considération d'une classe plus grande d'états quantiques dans l'espace de Fock ne satisfaisant pas nécessairement cette condition (**PI**). Cependant, la méthode développée dans [12] nécessite une condition de compacité dans l'interaction, donnée ci-dessous. Ainsi, l'approximation de champ moyen est effective dans l'espace de Fock sous les conditions de non perte de masse à l'infini

$$\exists \delta > 0\,, \exists C_\delta > 0\,, \forall \varepsilon \in \mathcal{E}\,, \qquad \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta < \infty\,,$$

et pour $\ell \in \{2, \ldots, r\}$, l'opérateur $\tilde{Q}_\ell$ est compact et autoadjoint dans $\bigvee^\ell \mathcal{Z}$.

On finira par le Chapitre 5 où l'on présentera en détails la preuve du principal résultat concernant la limite de champ dans un cadre abstrait d'un espace séparable d'Hilbert $\mathcal{Z}_0$ dans le cas de particules pigées sous les hypothèses (**A1**)-(**A2**)-(**C1**)-(**C2**)-(**D1**). Dans le cas non confiné, l'approximation de champ moyen sera validée sous les hypothèses (**A1**)-(**A2**)-(**C1**)-(**C2**)-(**D2**).
Les conditions (**A1**)-(**A2**) et (**C1**) correspondent respectivement à des conditions suffisantes d'existence de la dynamique quantique et au caractère globalement bien posé de l'équation (**Hartree**). Les hypothèses (**D1**)-(**D2**) ont été évoquées précédemment. Elles portent sur le noyau de la dérivée de l'interaction q et assurent la convergence vers l'équation (**Transport de mesures**). L'estimation (**C2**) correspond au contrôle de $v_t(z)$ permettant de résoudre l'équation (**Transport de mesures**) et d'établir l'unicité. On abordera dans la fin de ce Chapitre un résultat de convergence variationnel pour des particules confinées.

Chapter 2

Second Quantization and Wigner measures

The aim of this chapter is to introduce the second quantization formalism used throughout this thesis. So Fock spaces, creation, annihilation and Weyl operators will be recalled in finite and infinite dimensional Hilbert spaces. Many of this tools have been introduced by Berezin in [22] and a modern presentation of this subject can be found for instance in [26, 34, 16, 109, 114]. However, we will follow the point of view in [9, 10, 11, 12, 13, 6] which is suitable for the study of the mean-field problem since it provides a quantization map and an efficient symbolic calculus. In the first subsection 2.1.1, we introduce the Weyl, Wick and Anti-Wick quantization in finite dimension with some pros and cons of each quantization. The Bargmann transformation and Toeplitz correspondence are also discussed (see [11, 2, 102, 46] for example). The comparison of these quantizations uses the pseudo-differential calculus, see [64, 90, 25]. In subsection 2.1.2, the general definitions of Fock spaces and $\varepsilon-$dependant creation and annihilation operators are introduced while Wick quantization is discussed in subsection 2.1.3 with the class of symbol $\mathcal{P}_{p,q}$. In the last subsection 2.1.4; we define a wider class of symbols, denoted $\mathcal{Q}_{p,q}(A)$, and gather some of its main properties. The Wick quantization and the two classes $\mathcal{P}_{p,q}$, $\mathcal{Q}_{p,q}(A)$ are essential ingredients in proving the effectiveness of the mean-field approximation.

2.1 Creation, annihilation operators

2.1.1 Finite dimensional calculus

In this subsection we work in finite dimensional spaces. The creation and annihilation operators will be defined below. At first we introduce them in the space $\mathbb{C}^d$ and we will explain the construction of the Harmonic oscillator and the Number operator in this framework. In quantum mechanics, quantization of symbols is a fundamental question. We will define three quantizations (Weyl, Wick, Anti-Wick) in finite dimension and later in infinite dimensional Hilbert spaces. All of these quantizations have pros and cons and are equivalent in finite dimensional spaces.

We also consider coherent states in $\mathbb{R}^{2d}$ and emphasise their properties. After they will be generalized, in a Hilbert space in Section 2.1.3. At the end we briefly recall the so-called Bargmann representation and its correspondence with some quantized observables. We shall use the bracket notation of physicist: $|v\rangle$ will be the vector v while $\langle u|$ is the form $v \to \langle u, v\rangle$. For a normalized vector u, $|u\rangle\langle u|$ is the orthogonal projection on $\mathbb{C}u$. We shall work with a small parameter $h > 0$ or $\varepsilon > 0$. The rule of the semi-classical scaling will be summarized as follows

i) multiply any derivation by h, for any $\alpha \in \mathbb{N}^d$, $(h\partial_x)^\alpha = h^{|\alpha|}\partial_x^\alpha$ is a $0(1)$ operator;
ii) put a $\frac{1}{h}$ factor in any phase, $e^{i\frac{\varphi(x)}{h}}$ or $e^{\frac{\varphi}{h}}$.
iii) For integrations use the unit Lebesgue volume dx in the position variable and use $\frac{d\xi}{(2\pi h)^d}$ in the momentum variable.

Denote also the normalized Fourier transform on $\mathbb{R}^d$

$$[\mathcal{F}_h u](\xi) = \int_{\mathbb{R}^d} e^{-i\frac{\xi.x}{h}} u(x)dx\,,\ [\mathcal{F}_h^{-1} v](x) = \int_{\mathbb{R}^d} e^{i\frac{x.\xi}{h}} v(\xi)\frac{d\xi}{(2\pi h)^d}.$$

The Schwartz space will be denoted $\mathcal{S}(\mathbb{R}^d)$ and its dual, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. In the phase space, variables are denoted by capital letters: X, Y will be used for the variables $X = (x,\xi),\ Y = (y,\eta)$ in $\mathbb{R}^{2d}$. The symplectic form on $\mathbb{R}^{2d}$ will be denoted

$$\sigma(X,Y) = \xi.y - x.\eta = \sum_{j=1}^{d} \xi_j y_j - x_j\eta_j.$$

Definition 2.1.1. *For $X_0 = (x_0,\xi_0) \in \mathbb{R}^{2d}$, $\tau_{X_0}^h u := e^{i\frac{\xi_0\cdot(.-\frac{x_0}{2})}{h}} u(.-x_0)$ is called the phase translation of vector X_0. The function $\varphi_{X_0} := \frac{1}{(\pi h)^{\frac{d}{4}}}\tau_{X_0}^h\varphi_0$, with $\varphi_0(x) := \frac{1}{(\pi h)^{\frac{d}{4}}} e^{-\frac{x^2}{2h}}$, is the coherent state centered at X_0.*

Proposition 2.1.2. *1. For any $X_0 \in \mathbb{R}^{2d}$, $\tau_{X_0}^h$ is a unitary operator on $L^2(\mathbb{R}^d)$.*

2. For $X_1, X_2 \in \mathbb{R}^{2d}$, the Weyl relation holds

$$\tau_{X_1}^h \circ \tau_{X_2}^h = e^{i\frac{\sigma(X_1,X_2)}{2h}}\tau_{X_1+X_2}^h.$$

3. For any $X_0 \in \mathbb{R}^{2d},\ X_0 = (x_0,\xi_0)$

$$\tau_{X_0}^h = e^{i\frac{\xi_0\cdot x - x_0\cdot(hD_x)}{h}} = e^{i\frac{\sigma(X_0,(x,hD_x))}{h}},\ \textit{with } D_x = \frac{1}{i}\partial_x.$$

Remarks 2.1.3. *The Schwartz kernel of $\tau_{X_0}^h$ is given by*

$$\tau_{X_0}^h(x,y) = \int_{\mathbb{R}^d} e^{i\frac{\xi\cdot(x-y)}{h}} e^{i\frac{\xi_0\cdot(\frac{x+y}{2})-x_0\cdot\xi}{h}} \frac{d\xi}{(2h\pi)^d}\ .$$

Weyl quantization

We recall now the Weyl quantization for symbols in $\mathcal{S}'(\mathbb{R}^{2d})$. We collect some basic properties for these operators and we shall prove similar properties in infinite dimensional symplectic spaces.

Definition 2.1.4. *For any $b \in \mathcal{S}'(\mathbb{R}^{2d}_{x,y})$, the Weyl quantized operator $b^{Weyl}(x, hD_x) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is given by its kernel*

$$[b^{Weyl}(x, hD_x)](x, y) = \int_{\mathbb{R}^d} e^{i\frac{\xi.(x-y)}{h}} b(\frac{x+y}{2}, \xi) \frac{d\xi}{(2h\pi)^d}.$$

Proposition 2.1.5. *1. Any $B \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ has a Weyl symbol*

$$B_{Weyl}(r, \xi) = \int_{\mathbb{R}^d} e^{i\frac{\xi.(x-y)}{h}} B(r + \frac{s}{2}, r - \frac{s}{2})\ ds.$$

2. *The formal adjoint of $b^{Weyl}(x, hD_x)$ is $\bar{b}^{Weyl}(x, hD_x)$.*

3. *If $b_1, b_2 \in L^2(\mathbb{R}^{2d})$, the operators $b_1^{Weyl}(x, hD_x)$, $b_2^{Weyl}(x, hD_x)$ are Hilbert-Schmidt operators and*

$$\mathrm{Tr}\ [b_1^{Weyl}(x, hD_x)^* b_2^{Weyl}(x, hD_x)] = \int_{\mathbb{R}^{2d}} \overline{b_1(x, \xi)} b_2(x, \xi)\ dx \frac{d\xi}{(2\pi h)^d}.$$

4. *The conjugation with respect to phase translation centered in $X_0 \in \mathbb{R}^{2d}$ yields*

$$\tau^h_{X_0} b^{Weyl}(x, hD_x) \tau^h_{-X_0} = [b(. - X_0)^{Weyl}(x, hD_x)],$$

for any $b \in \mathcal{S}'(\mathbb{R}^{2d})$.

Anti-Wick quantization

Take $X_0 \in \mathbb{R}^{2d}$, and set the operator $\Pi^h_{X_0} = |\varphi_{X_0}\rangle\langle\varphi_{X_0}|$, where φ_{X_0} is the coherent state centered at X_0. The following Proposition defines the Anti-Wick quantization for symbols in $\mathcal{S}'(\mathbb{R}^{2d})$. One of the pros of this quantization is its positivity. The link with the Weyl quantization is specified in Proposition 2.1.8 below.

Proposition 2.1.6. *For any $b \in \mathcal{S}'(\mathbb{R}^{2d})$ the operator*

$$b^{A-Wick}(x, hD_x) = \int_{\mathbb{R}^{2d}} b(X_0) \Pi^h_{X_0} \frac{dX_0}{(2\pi h)^d}, \tag{2.1.1}$$

is well-defined and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Moreover the following equivalence holds

$$b^{A-Wick}(x, hD_x) = c^{Weyl}(x, hD_x) \Leftrightarrow c = b * (\frac{e^{-|x|^2}}{(h\pi)^d}). \tag{2.1.2}$$

Remarks 2.1.7. *The Anti-Wick quantization is positive, i.e.,*

$$b \geqslant 0 \Longrightarrow b^{A-Wick}(x, hD_x) \geqslant 0.$$

Proposition 2.1.8. *For any continuous operator $B:\ \mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ the symbol*

$$\sigma^{Wick}(B)(X_0)=\langle\varphi_{X_0},B\,\varphi_{X_0}\rangle,$$

is well-defined. For $B=b^{Weyl}(x,hD_x)$, one has

$$\sigma^{Wick}(b^{Weyl}(x,hD_x))=b*(\frac{e^{-|\cdot|^2}}{(h\pi)^d}),$$

and the following positivity property holds

$$(B\geqslant 0)\Longrightarrow(\sigma^{Wick}(B))\geqslant 0.$$

We use the notation $\mathcal{L}(\mathfrak{h})$ for the space of bounded operators on $\mathfrak{h}$ and $\mathcal{L}^p(\mathfrak{h})$, $1\leqslant p\leqslant+\infty$, for the Schatten classes, $\mathcal{L}^\infty(\mathfrak{h})$ being the space of compact operators for $p=+\infty$.

Remarks 2.1.9. **i)** *If $B\in\mathcal{L}^1(L^2(\mathbb{R}^d))$, then $\sigma^{Wick}(B)\in L^1(\mathbb{R}^{2d})$ and*

$$\mathrm{Tr}\,[B]=\int_{\mathbb{R}^{2d}}\sigma^{Wick}(B)(X)\frac{dX}{(2\pi h)^d}.$$

ii) *We cannot define the Wick quantization for any symbol in $\mathcal{S}'(\mathbb{R}^{2d})$ but it works for polynomials, and it is the subject of the next paragraph. When it makes sense, the correspondence between the Wick and Anti-Wick symbol is given by*

$$\mathrm{Tr}\,[b^{A-Wick}C]=\int b(X)\sigma^{Wick}(C)(X)\frac{dX}{(2\pi h)^d}.$$

Creation, Annihilation operators and Wick quantization

Let introduce the creation, annihilation operators in the finite dimensional space $\mathbb{C}^d$. Denote $(e_1,...,e_d)$ its canonical orthonormal basis and set

$$a(e_j)=a_j=(h\partial_{x_j}+x_j)\ \ ;\ \ a^*(e_j)=a_j^*=(-h\partial_{x_j}+x_j)\tag{2.1.3}$$

$$\forall g=(g_1,...,g_d)\in\mathbb{C}^d,\ a(g)=\sum_{j=1}^d\bar{g}_ja_j\ ,\ a^*(g)=\sum_{j=1}^d g_ja_j^*.\tag{2.1.4}$$

The operators $a(g),a^*(g)$ satisfy the canonical commutation relations (CCR). For any $f,g\in\mathbb{C}^d$

$$[a(g),a(f)]=a(g)a(f)-a(f)a(g)=[a^*(g),a^*(f)]=0,\tag{\textbf{CCR}}$$

$$[a(g),a^*(f)]=\varepsilon\langle g,f\rangle\ \mathrm{Id}\ ,\ \varepsilon=2h.$$

The vector φ_0 introduced earlier is also denoted Ω and called the vacuum. The harmonic oscillator is denoted $\mathbf{N}_d$ and given by the formula

$$\mathbf{N}_d:=(-h^2\Delta+x^2-hd)=\sum_{j=1}^d a_j^*a_j.\tag{\textbf{Number operator}}$$

We can construct an orthonormal basis of eigenfunctions called normalized α^{th} Hermite functions Ψ_α for any multi-index $\alpha \in \mathbb{N}^d$ given by

$$\Psi_\alpha = \frac{1}{\sqrt{\varepsilon^{|\alpha|}\alpha!}}(a^*)^\alpha|\Omega\rangle. \qquad \textbf{(Hermite functions)}$$

The notation $(a^*)^\alpha := \prod_{j=1}^d (a^*(e_j))^{\alpha_j}$ for a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$ is correct since $a_j^* = a(e_j)^*$ commute.

Introduce now the field operator on $\mathbb{C}^d$ given for any $g \in \mathbb{C}^d$, by $\phi(g) = \frac{1}{\sqrt{2}}(a^*(g) + a(g))$. Then the unitary group generated by the field operator $\phi(g)$ defines the so-called Weyl operator

$$\mathcal{W}(f) = e^{i\phi(f)}. \qquad \textbf{(Weyl operators)}$$

Moreover the Weyl operator $\mathcal{W}(f)$ is equal to $\tau^h_{h\sqrt{2}X_0}$ with $X_0 = (x_0, \xi_0)$ and $f = \frac{1}{i}(x_0 + i\xi_0)$. So, the Weyl relations follow

$$\forall f_1, f_2 \in \mathbb{C}^d,\ \mathcal{W}(f_1) \circ \mathcal{W}(f_2) = e^{-i\varepsilon\frac{\operatorname{Im}\langle f_1, f_2\rangle}{2}}\mathcal{W}(f_1 + f_2),$$

$$\mathcal{W}^*(f)a(g)\mathcal{W}(f) = a(g) + \frac{i\varepsilon}{\sqrt{2}}\langle g, f\rangle.$$

The function $E(z) := \mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle$ satisfies the equality for any $z \in \mathbb{C}^d$

$$\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle = e^{\frac{a^*(z)-a(z)}{\varepsilon}}|\Omega\rangle = \tau_z^h\varphi_0,\ z = (z_R, z_I).$$

Now we introduce the Wick quantization for polynomials on $\mathbb{R}^{2d}_{x,\xi}$ identified with $\mathbb{C}^d$.

Definition 2.1.10. *For any polynomial $P(z,\bar{z}) = \sum_{|\alpha|+|\beta|\leqslant n} c_{\alpha,\beta}\bar{z}^\alpha z^\beta$ on $\mathbb{R}^{2d}_{x,\xi}$ identified with $\mathbb{C}^d$ via $z = x + i\xi$, its Wick quantization is given by*

$$P^{Wick}(z,\bar{z}) = \sum_{|\alpha|+|\beta|\leqslant n} c_{\alpha,\beta}(a^*)^\alpha a^\beta. \tag{2.1.5}$$

Remarks 2.1.11. *The Wick rule consists by replacing z (resp $\bar{z}$) by a (resp a^*) and keeping all the annihilation operators on the right-hand side.*

Bargmann representation and Toeplitz correspondance

The aim of this paragraph is to introduce the Bargmann transformation. Actually this transformation is useful to deal with the so-called Lowest Landau Level (LLL) model used in Chapter 5, Example 5.2.11. We will give correspondence between some Wick symbols and Toeplitz operators.

Definition 2.1.12. *For a function $u \in \mathcal{S}'(\mathbb{R}^d)$, the Bargmann transform is given by*

$$[B_h u](z) = \frac{1}{(\pi h)^{\frac{3d}{4}}}\, e^{\frac{z^2}{2h}}\int_{\mathbb{R}^d} e^{-\frac{(\sqrt{2}z-y)^2}{2h}}u(y)dy,\ \ z = \frac{x - i\xi}{\sqrt{2}} \in \mathbb{C}^d, \tag{2.1.6}$$

where $h > 0$ is a small parameter.

There are a lot of similar definitions for the Bargmann-Fock-Segal transformation. For example in [46] the Bargmann B transformation is given by the formula

$$[Bu](z) = 2^{\frac{d}{4}} \int_{\mathbb{R}^d} u(x) e^{2\pi xz - \pi x^2 - \frac{\pi}{2} z^2} dx.$$

The Bargmann space $\mathcal{F}_h$ is given by

$$\mathcal{F}_h = \{ f \in L^2(\mathbb{C}^d, e^{-\frac{|z|^2}{2h}} \frac{L(dz)}{(2\pi h)^d}),\ \partial_{\bar{z}} f = 0\},$$

and is equipped with the norm,

$$||f||^2_{\mathcal{F}_h} = \int_{\mathbb{C}^d} |f(z)|^2 \, e^{-\frac{|z|^2}{2h}} \, \frac{L(dz)}{(2\pi h)^d} \,,$$

and it is a closed subspace of $L^2(\mathbb{C}^d, e^{-\frac{|z|^2}{2h}} \frac{L(dz)}{(2\pi h)^d})$ with a related orthogonal projection Π_h given explicitly by

$$\begin{aligned} \Pi_h : L^2(\mathbb{C}^d, e^{-\frac{|.|^2}{2h}} \frac{L(d\xi)}{(2\pi h)^d}) \to \mathcal{F}_h \\ g \mapsto \Pi_h(g)(z) := \int_{\mathbb{C}^d} e^{\frac{z.\bar{\tau} - |\tau|^2}{2h}} \, g(\tau) \, \frac{L(d\tau)}{(2\pi h)^d}. \end{aligned} \tag{2.1.7}$$

Proposition 2.1.13. *The Bargmann transform B_h is a unitary map from $L^2(\mathbb{R}^d, dx)$ into $\mathcal{F}_h$ satisfying $B_h^* B_h = \mathrm{Id}$ and $B_h B_h^* = \Pi_h$.*

Remarks 2.1.14. *Instead of considering holomorphic functions (replace z by $\bar{z}$ in the definition of B_h), one could consider anti-holomorphic functions. The Anti-Wick quantization of a polynomial $P(z, \bar{z}) = \sum_{|\alpha|+|\beta| \leqslant n} c_{\alpha,\beta} \bar{z}^\alpha z^\beta$ is obtained by replacing z(resp $\bar{z}$) by a (resp by a^*) while keeping a in the left-hand side.*

Examples 2.1.15. *We enumerate some examples in relation with the creation and annihilation operators where $\varepsilon = 2h$.*

1. *The image of Hermite functions Ψ_α by the Bargmann transformation is given by polynomials $\frac{1}{\sqrt{\varepsilon^\alpha \alpha!}} |z|^\alpha$.*

2. *The image of coherent functions φ_{z_0} by the Bargmann transformation is given by $e^{-\frac{|z_0|^2}{2\varepsilon}} e^{\frac{\langle \bar{z}_0, z\rangle}{\varepsilon}}$.*

3. *The annihilation operator $a_j = (h\partial_{x_j} + x_j)$ satisfies $\left[B_h(a_j)B_h^*\right]_{|\mathcal{F}_h \to \mathcal{F}_h} = \varepsilon \partial_{z_j} = \Pi_h(\bar{z}_j \times)\Pi_h$.*

4. *The creation operator $a_j^* = (-h\partial_{z_j} + x_j)$ satisfies $\left[B_h(a_j^*)B_h^*\right]_{|\mathcal{F}_h \to \mathcal{F}_h} = z_j \times$.*

5. *When $P(z, \bar{z}) = \sum_{|\alpha|+|\beta| \leqslant n} c_{\alpha,\beta} z^\alpha \bar{z}^\beta$ is a polynomial on $\mathbb{R}^d$ identified on $\mathbb{C}^d$ via $z = x + i\xi$, the Anti-Wick quantization is given by $P^{A-Wick}(x, hD_x) = B_h^*(P\times)B_h$. Thus*

$$B_h P^{A-Wick} B_h^* = \Pi_h(P\times)\Pi_h$$

is a Toeplitz operator.

2.1.2 The Fock space

First definitions

Let $\mathcal{Z}$ be a separable Hilbert space with the inner product $\langle . , . \rangle$ antilinear on the left-hand side associated with a norm $|z| = \sqrt{\langle z , z \rangle}$. The purpose of this section is to generalize some tools introduced above to a more general framework. The semi-classical parameter will be denoted ε. If not specified tensor products and orthogonal direct sums are considered in their Hilbert completed version.

Definition 2.1.16. *The bosonic Fock space on $\mathcal{Z}$ is given by:*

$$\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \bigvee^{n} \mathcal{Z}, \qquad \textbf{(Fock space)}$$

where $\bigvee^n \mathcal{Z}$ denotes the n-fold symmetric tensor product. For all $n \in \mathbb{N}$ the orthogonal projection of $\bigotimes^n \mathcal{Z}$ on the subspace $\bigvee^n \mathcal{Z}$ is denoted by $\mathcal{S}_n$. Moreover $\mathcal{S}_n$ have the explicit writing, for all $\xi_1, ..., \xi_n \in \mathcal{Z}$:

$$\xi_1 \vee \xi_2 \vee ... \vee \xi_n = \mathcal{S}_n(\xi_1 \otimes \xi_2 \otimes ... \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \otimes ... \otimes \xi_{\sigma(n)},$$

where Σ_n is the n-th fold symmetric group.

A useful dense subspace of $\Gamma_s(\mathcal{Z})$ is the algebraic direct sum $\Gamma_s^{fin}(\mathcal{Z}) := \bigoplus_{n=0}^{alg} \bigvee^n \mathcal{Z}$.

Proposition 2.1.17. *The family $(\xi_1 \vee \xi_2 \vee ... \vee \xi_n)_{\xi_i \in \mathcal{Z}, i=1,...,n}$ spans $\bigvee^{n,alg} \mathcal{Z}$ and is a total family of $\bigvee^n \mathcal{Z}$. The same property holds for $(z^{\otimes n})_{z \in \mathcal{Z}, n \in \mathbb{N}}$.*

Proof. $\mathcal{S}_n$ is an orthogonal projection since $\bigvee^n \mathcal{Z}$ is a closed subspace, then the family $(\xi_1 \vee \xi_2 \vee ... \vee \xi_n)_{\xi_i \in \mathcal{Z}}$ spans $\bigvee^{n,alg} \mathcal{Z}$ and is a total family of $\bigvee^n \mathcal{Z}$. The last result is straightforward from the polarization identity

$$\xi_1 \vee \xi_2 \vee ... \vee \xi_n = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 ... \varepsilon_n \Big(\sum_{j=1}^{n} \varepsilon_j \xi_j \Big)^{\otimes n}.$$

∎

For $k = 1, 2$ and any operators $A_k : \bigvee^{i_k} \mathcal{Z} \to \bigvee^{j_k} \mathcal{Z}$ we can define the symmetric tensor product of operators $A_1 \vee A_2$ by

$$A_1 \vee A_2 = S_{j_1+j_2} \circ (A_1 \otimes A_2) \circ S_{i_1+i_2} \in \mathcal{L}\Big(\bigvee^{i_1+i_2} \mathcal{Z}, \bigvee^{j_1+j_2} \mathcal{Z}\Big).$$

For all $z \in \mathcal{Z}$, recall that $|z\rangle$ is the operator: $\lambda \in \mathbb{C} \mapsto \lambda z \in \mathcal{Z}$ and $\langle z|$ is the linear functional: $\xi \mapsto \langle z, \xi \rangle \in \mathbb{C}$. Now let introduce the annihilation and creation operators in a infinite dimensional Hilbert space and generalize the Definition 2.1.3.

Definition 2.1.18. *For $z \in \mathcal{Z}$, $n \in \mathbb{N}$ and $\varepsilon > 0$ a parameter. The ε-dependent annihilation and creation operators are defined by the following equalities*

$$a(z)_{|\bigvee^{n+1}\mathcal{Z}} = \sqrt{\varepsilon(n+1)}\langle z| \otimes \text{ Id }_{|\bigvee^{n}\mathcal{Z}},$$
$$a^*(z)_{|\bigvee^{n}\mathcal{Z}} = \sqrt{\varepsilon(n+1)}\mathcal{S}_{n+1} \circ (|z\rangle \otimes \text{ Id }_{|\bigvee^{n}\mathcal{Z}}) = \sqrt{\varepsilon(n+1)}\ |z\rangle \vee \text{ Id }_{|\bigvee^{n}\mathcal{Z}}.$$

The families $(a(z))_{z\in\mathcal{Z}}$ and $(a^*(z))_{z\in\mathcal{Z}}$ satisfy the canonical commutation relations for all $z_1, z_2 \in \mathcal{Z}$:

$$[a(z_1), a^*(z_2)] = \varepsilon\langle z_1, z_2\rangle \text{ Id }, \quad [a(z_1), a(z_2)] = 0, \quad [a^*(z_1), a^*(z_2)] = 0.$$

We also consider another important operator, namely the field operator

$$\phi(z) = \frac{1}{\sqrt{2}}(a^*(z) + a(z)), \qquad \textbf{(Field operators)}$$

generator of the unitary group $\mathcal{W}(z) = e^{i\phi(z)}$ which satisfies the Weyl commutation relations for all $z_1,\ z_2 \in \mathcal{Z}$,

$$\mathcal{W}(z_1)\mathcal{W}(z_2) = e^{-\frac{i\varepsilon}{2}\text{ Im }\langle z_1, z_2\rangle}\mathcal{W}(z_1 + z_2)\ .$$

When $\mathcal{Z} = \mathbb{C}^d$ the definition coincides with the one introduced earlier in (**Weyl operators**). The generating functional associated with this representation is given by

$$\langle\Omega, \mathcal{W}(z)\Omega\rangle = e^{-\frac{\varepsilon}{4}|z|^2},$$

where Ω is the vacuum vector $(1, 0, ...) \in \Gamma_s(\mathcal{Z})$. The total family of coherent vectors $E(z) = \mathcal{W}(\frac{\sqrt{2}z}{i\varepsilon})\Omega$, has the explicit form

$$E(z) = e^{-\frac{|z|^2}{2\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{\varepsilon^n} \frac{a^*(z)^n}{n!}\Omega = e^{-\frac{|z|^2}{2\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^{-\frac{n}{2}} \frac{z^{\otimes n}}{\sqrt{n!}}. \qquad \textbf{(Coherent states)}$$

The number operator $\mathbf{N}$, parametrized by $\varepsilon > 0$, is defined according to

$$\mathbf{N}_{|\vee^n\mathcal{Z}} = \varepsilon n \text{ Id }_{\vee^n\mathcal{Z}}. \qquad \textbf{(Number operator)}$$

The second quantized operator can be defined by the formula

$$d\Gamma(A)_{|\vee^{n,alg}D(A)} = \varepsilon \sum_{k=1}^{n} \text{ Id }^{\otimes(k-1)} \otimes A \otimes \text{ Id }^{\otimes(n-k)}. \qquad \textbf{(Second quantized operator)}$$

In particular, $\mathbf{N} = d\Gamma(\text{ Id })$. We can also define the operator Γ. Let $\mathcal{Z}_i,\ i = 1, 2$ be two Hilbert spaces. Denote $q:\ \mathcal{Z}_1 \mapsto \mathcal{Z}_2$ a bounded linear operator. Then the Gamma operator is given by

$$\Gamma(q) : \Gamma(\mathcal{Z}_1) \mapsto \Gamma(\mathcal{Z}_2)$$
$$\Gamma(q)_{|\bigvee^n\mathcal{Z}_1} = q \otimes ... \otimes q.$$

One of the first properties of this functor Γ is its link with the second quantized operator $d\Gamma$ given by the formula for $A \in \mathcal{L}(\mathcal{Z})$,

$$e^{d\Gamma(A)} = \Gamma(e^A). \qquad \textbf{(free Hamiltonian)}$$

Tensor product of Fock spaces

Consider $\mathcal{Z}_i,\ i = 1,2$ two Hilbert spaces. Let i_1, i_2 the injections of $\mathcal{Z}_1, \mathcal{Z}_2$ into $\mathcal{Z}_1 \oplus \mathcal{Z}_2$. We define $U:\ \Gamma(\mathcal{Z}_1) \otimes \Gamma(\mathcal{Z}_2) \to \Gamma(\mathcal{Z}_1 \oplus \mathcal{Z}_2)$ as follows

$$Uu \otimes v := \sqrt{\frac{(p+q!)}{p!q!}}\Gamma(i_1)u \bigvee \Gamma(i_2)v,\quad u \in \bigvee^{p} \mathcal{Z}_1,\ v \in \bigvee^{q} \mathcal{Z}_2. \tag{2.1.8}$$

The operator U is unitary. The following Proposition will be very useful in bosonic QFT.

Proposition 2.1.19. *Assume that the Hilbert space $\mathcal{Z}$ can be decomposed into a direct sum of Hilbert spaces $\mathcal{Z}_i,\ i = 1,2$ such that $\mathcal{Z} = \mathcal{Z}_1 \overset{\perp}{\oplus} \mathcal{Z}_2$. Then we have, though U, the canonical identification*

$$\Gamma_s(\mathcal{Z}) \cong \Gamma_s(\mathcal{Z}_1) \otimes \Gamma_s(\mathcal{Z}_2).$$

Remarks 2.1.20. *With this decomposition we have*

1. $|\Omega\rangle = |\Omega_1\rangle \otimes |\Omega_2\rangle$.
2. *For* $z_1 \in \mathcal{Z}_1,\ z_2 \in \mathcal{Z}_2$,
$$a(z_1) = a_1(z_1) \otimes \text{ Id }_{\Gamma_s(\mathcal{Z}_2)} \quad , \quad a^*(z_1) \otimes \text{ Id }_{\Gamma_s(\mathcal{Z}_2)}$$
$$\mathbf{N} = \mathbf{N_1} \otimes \text{ Id }_{\Gamma_s(\mathcal{Z}_2)} + \text{ Id }_{\Gamma_s(\mathcal{Z}_1)} \otimes \mathbf{N_2}.$$
3. *Take* $\Psi_1 \in \mathcal{Z}_1$ *and* $\Psi_2 \in \mathcal{Z}_2$ *with* $|\Psi_l|_{\mathcal{Z}_l} = 1,\ N_1, N_2 \in \mathbb{N}$, *and set* $\varrho_l = |\Psi_l^{\otimes N_l}\rangle\langle\Psi_l^{\otimes N_l}|$ *for* $l = 1,2$. *The tensor states* $\varrho_1 \otimes \varrho_2$ *is the twin states*
$$|\Psi_1^{\otimes N_1} \otimes \Psi_2^{\otimes N_2}\rangle\langle\Psi_1^{\otimes N_1} \otimes \Psi_2^{\otimes N_2}| \tag{Twin states}$$
in $\Gamma_s(\mathcal{Z}_1) \otimes \Gamma_s(\mathcal{Z}_2)$. *Within the tensor decomposition we can identify the vector* $\Psi^{\vee(N_1,N_2)} \in \Gamma_s(\mathcal{Z})$ *associated with* $\Psi_1^{\otimes N_1} \otimes \Psi_2^{\otimes N_2}$. *More precisely the following writing holds*
$$\begin{aligned}\Psi_l^{\otimes N_l} &= \frac{1}{\sqrt{\varepsilon^{N_l}N_l!}}a^*(\Psi_l)...a^*(\Psi_l)|\Omega_l\rangle,\ l = 1,2\\ \Psi^{\vee(N_1,N_2)} &= \sqrt{\frac{(N_1+N_2)!}{\varepsilon^{N_1+N_2}N_1!N_2!}}\mathcal{S}_{N_1+N_2}(\Psi_1^{\otimes N_1} \otimes \Psi_2^{\otimes N_2})\\ &= \frac{1}{\sqrt{\varepsilon^{N_1+N_2}N_1!N_2!}}\underbrace{a^*(\Psi_1)...a^*(\Psi_1)}_{N_1 \text{ times}}\underbrace{a^*(\Psi_2)...a^*(\Psi_2)}_{N_2 \text{ times}}|\Omega\rangle \textit{ in } \Gamma_s(\mathcal{Z})\end{aligned}$$

Example 2.1.21.

1. $\mathbb{C}^{d_1+d_2} = \mathbb{C}^{d_1} \overset{\perp}{\oplus} \mathbb{C}^{d_2}$, *then* $L^2(\mathbb{R}^{d_1+d_2}, dx) = L^2(\mathbb{R}^{d_1}, dx) \otimes L^2(\mathbb{R}^{d_2}, dx)$.
2. *Denote* $\mathfrak{p}$ *a finite rank orthogonal projection on a Hilbert space* $\mathcal{Z}$, *then*
$$\Gamma_s(\mathcal{Z}) = \Gamma_s(\mathfrak{p}\mathcal{Z}) \otimes \Gamma_s((1-\mathfrak{p})\mathcal{Z}).$$
3. *When* $\mathcal{Z} = \mathcal{Z}_1 \overset{\perp}{\oplus} \mathcal{Z}_2$, *then for any* $z = z_1 + z_2$, $\mathcal{W}(z) = \mathcal{W}_{\mathcal{Z}_1}(z) \otimes \mathcal{W}_{\mathcal{Z}_2}(z_2)$.

2.1.3 Wick quantization

In this subsection, we introduce the Wick symbolic calculus for polynomials with bounded kernels. It is related to the normal ordering of products of creation-annihilation operators which is a well treated subject in standard textbooks (see for instance [22, 35]). Here we follow the presentation in [9] which stresses the symbol-operator correspondence and which is more convenient for our purpose, especially in Section 4. We will extend the Wick quantization to some classes of polynomials with unbounded kernels in Subsection 2.1.4. For all p,q $\in \mathbb{N}$, we denote $\mathcal{P}_{p,q}(\mathcal{Z})$ the space of complex-valued polynomials on $\mathcal{Z}$, defined by the following continuity condition

$$b \in \mathcal{P}_{p,q}(\mathcal{Z}) \Leftrightarrow \exists \tilde{b} \in \mathcal{L}(\bigvee^{p} \mathcal{Z}, \bigvee^{q} \mathcal{Z}), b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle. \tag{2.1.9}$$

These spaces are equipped with norms $|.|_{\mathcal{P}_{p,q}}$:

$$|b|_{\mathcal{P}_{p,q}} = \|\tilde{b}\|_{\mathcal{L}(\bigvee^{p} \mathcal{Z} \bigvee^{q} \mathcal{Z})}.$$

The subspace of $\mathcal{P}_{p,q}(\mathcal{Z})$ polynomials b such that $\tilde{b}$ is a compact operator is denoted by $\mathcal{P}^{\infty}_{p,q}(\mathcal{Z})$. Later we will see that this subspace plays a important role in the convergence towards Wigner measures.

Definition 2.1.22. *For each symbol $b(z) \in \mathcal{P}_{p,q}(\mathcal{Z})$, is associated an operator: $\Gamma_s^{fin}(\mathcal{Z}) \longrightarrow \Gamma_s^{fin}(\mathcal{Z})$, given by*

$$b^{Wick}_{|\bigvee^n \mathcal{Z}} = 1_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} \varepsilon^{\frac{p+q}{2}} \mathcal{S}_{n-p+q}(\tilde{b} \otimes \text{ Id }^{\otimes(n-p)}) \,.$$

The Wick quantization map depends in the parameter $\varepsilon > 0$, however for simplicity we omit this dependence in the notation of b^{Wick}. By linearity one can extend this quantization to any finite sum in $\mathcal{P}_{alg}(\mathcal{Z}) := \oplus^{alg}_{p,q \geqslant 0} \mathcal{P}_{p,q}(\mathcal{Z})$. Many examples of Wick observables have been introduced in previous subsection. In fact we have the following correspondance on $\Gamma_s^{fin}(\mathcal{Z})$ for $\xi \in \mathcal{Z}$ and $A \in \mathcal{L}(\mathcal{Z})$:

$$\begin{aligned} \langle z, \xi \rangle^{Wick} &= a^*(\xi) \;,\; \langle \xi, z \rangle^{Wick} = a(\xi) \\ (|z|^2)^{Wick} &= \mathbf{N} \;,\; \langle z, Az \rangle^{Wick} = d\Gamma(A). \end{aligned}$$

Remarks 2.1.23. *Sometimes we consider the Wick quantization of a symbol b, or equivalently the Wick quantization of its kernel $\tilde{b}$. Moreover owing to the condition $\tilde{b} \in \mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$ for a symbol $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the definition implies that any Gateaux-differential $\partial^j_{\bar{z}} \partial^k_z b(z)$ at $z \in \mathcal{Z}$ equals*

$$\partial^j_{\bar{z}} \partial^k_z b(z) = \frac{p!}{(p-k)!} \frac{q!}{(q-j)!} (\langle z^{\otimes q-j} | \vee \text{ Id }_{|\bigvee^j \mathcal{Z}}) \, \tilde{b} \, (|z^{\otimes p-k} \rangle \vee \text{ Id }_{|\bigvee^k \mathcal{Z}}) \in \mathcal{L}(\bigvee^{k} \mathcal{Z}, \bigvee^{j} \mathcal{Z}). \tag{2.1.10}$$

Additionally the vector space spanned by all these Wick polynomials will be denote $\mathcal{P}$.

Proposition 2.1.24. *The following identities hold true on $\Gamma_s^{fin}(\mathcal{Z})$ for every $b \in \mathcal{P}_{p,q}(\mathcal{Z})$*

1. $(b^{Wick})^* = \overline{b(z)}^{Wick}$,
2. $(C(z)b(z)A(z))^{Wick} = C^{Wick} b^{Wick} A^{Wick}$, *if $A \in \mathcal{P}_{\alpha,0}(\mathcal{Z})$, $C \in \mathcal{P}_{0,\beta}(\mathcal{Z})$.*

Proposition 2.1.25. *1. The Wick operator associated with $b(z) = \prod_{i=1}^{p} \langle z, \eta_i \rangle \prod_{j=1}^{q} \langle \xi_j, z \rangle$, $\eta_i, \xi_j \in \mathcal{Z}$ equals*

$$b^{Wick} = a^*(\eta_1)...a^*(\eta_p)a(\xi_1)...a(\xi_p).$$

2. For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ and $z \in \mathcal{Z}$ the following equality holds true

$$\langle z^{\otimes j}, b^{Wick} z^{\otimes k} \rangle = \delta^+_{k-p,j-q} \sqrt{\frac{k!j!}{(k-p)!(j-q)!}} \varepsilon^{\frac{p+q}{2}} |z|^{k-p+j-q} \ b(z),$$

holds for any $k, j \in \mathbb{N}$. The symbol $\delta^+_{\alpha,\beta}$ denotes $\delta_{\alpha,\beta} 1_{[0,+\infty)}(\alpha)$ where $\delta_{\alpha,\beta}$ is the standard Kronecker symbol.

A consequence of 1. is that when $b \in \mathcal{P}_{p,p}(\mathcal{Z})$ and $\tilde{b} \geqslant 0$, then $b^{Wick} \geqslant 0$ on $\Gamma_s^{fin}(\mathcal{Z})$. However, it is not true in general. When the Hilbert space $\mathcal{Z}$ equals $L^2(\mathbb{R}^d, dx)$, the general formula for b^{Wick} with $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ is

$$b^{Wick} = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(y_1, ..., y_q, x_1, ..., x_p) a^*(y_1)...a^*(y_q) a(x_1)...a(x_p) \ dx_1...dx_p dy_1...dy_q,$$

where $\tilde{b}(y, x)$ is the Schwartz kernel of $\tilde{b}$ and where $a(x_k) = a(\delta_{x_k})$. The following Proposition is the so-called Number estimates. It plays a fundamental role to control the unboundeness of some Wick observables in the Fock space with the help of the Number operator $\mathbf{N}$ introduced in Definition (**Number operator**).

Proposition 2.1.26. *1. For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, and for any $k, j \in \mathbb{N}$*

$$|b^{Wick}|_{\mathcal{L}(\bigvee^k \mathcal{Z}, \bigvee^j \mathcal{Z})} \leqslant \delta^+_{k-p,j-q} (j\varepsilon)^{\frac{q}{2}} (k\varepsilon)^{\frac{p}{2}} |\tilde{b}|_{\mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})}.$$

This implies the Number estimate

2. For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the estimate

$$\|\langle \mathbf{N} \rangle^{-\frac{q}{2}} b^{Wick} \langle \mathbf{N} \rangle^{-\frac{p}{2}} \|_{\mathcal{L}(\Gamma_s(\mathcal{Z}))} \leqslant |b|_{\mathcal{P}_{p,q}}, \qquad \textbf{(Number estimate)}$$

holds with $\langle \mathbf{N} \rangle = (1 + \mathbf{N}^2)^{\frac{1}{2}}$. Moreover, if $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, then $\langle \mathbf{N} \rangle^{-\frac{p+q}{2}} b^{Wick}$ and $b^{Wick} \langle \mathbf{N} \rangle^{-\frac{p+q}{2}}$ can be extended to bounded operators on $\Gamma_s(\mathcal{Z})$ with norms smaller than $C_{p,q}|b|_{\mathcal{P}_{p,q}}$, for some constants $C_{p,q}$.

An important operation with the Wick symbols is the composition: $b_1^{Wick} \circ b_2^{Wick}$ with b_1,$b_2 \in \mathcal{P}_{alg}(\mathcal{Z})$ turns to be a Wick symbol in $\mathcal{P}_{alg}(\mathcal{Z})$. Now we introduce the useful notations for the formula about the composition.
Let $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the k-th differential of b is well defined and

$$\partial_z^k b(z) \in (\bigvee^k \mathcal{Z})^*, \quad \text{and} \quad \partial_{\bar{z}}^k b(z) \in \bigvee^k \mathcal{Z}.$$

We use the following notations about the Poisson brackets:

$$\{b_1, b_2\}^{(k)}(z) = \partial_z^k b_1(z).\partial_{\bar{z}}^k b_2(z) - \partial_z^k b_2(z).\partial_{\bar{z}}^k b_1(z).$$

with the $\mathbb{C}-$bilinear duality product $\partial_z^k b_1(z).\partial_{\bar{z}}^k b_2(z) = \langle \partial_z^k b_1(z), \partial_{\bar{z}}^k b_2(z) \rangle_{((\bigvee^k \mathcal{Z})^*, \bigvee^k \mathcal{Z})}$,
which defines a function of $z \in \mathcal{Z}$ simply denoted by $\partial_z^k b_1.\partial_{\bar{z}}^k b_2$.

Proposition 2.1.27. *Let $b_1 \in \mathcal{P}_{p_1,q_1}(\mathcal{Z})$ et $b_2 \in \mathcal{P}_{p_2,q_2}(\mathcal{Z})$. For all $k \in \{0,...,\min(p_1,q_2)\}$, $\partial_z^k b_1.\partial_{\bar z}^k b_2$ belongs to $\mathcal{P}_{p_1+p_2-k,q_1+q_2-k}(\mathcal{Z})$, we have the estimate*

$$|\partial_z^k b_1.\partial_{\bar z}^k b_2|_{\mathcal{P}_{p_1+p_2,q_1+q_2}} \leqslant \frac{p_1!}{(p_1-k)!}\frac{q_2!}{(q_2-k)!}|b_1|_{\mathcal{P}_{p_1,q_1}}|b_2|_{\mathcal{P}_{p_2,q_2}},$$

and the following formulas hold true on $\Gamma_s^{fin}(\mathcal{Z})$:

1.
$$b_1^{Wick}\circ b_2^{Wick} = \Big[\sum_{k=0}^{\min(p_1,q_2)}\frac{\varepsilon^k}{k!}\partial_z^k b_1.\partial_{\bar z}^k b_2\Big]^{Wick}.$$

2.
$$[b_1^{Wick}, b_2^{Wick}] = \sum_{k=1}^{\max(\min(p_1,q_2),\min(p_2,q_1))}\frac{\varepsilon^k}{k!}\Big[\{b_1,b_2\}^{(k)}\Big]^{Wick}.$$

Let us finish this subsection by some regularity properties on Wick observables with respect to (**Number operator**), (**Weyl operators**), or the (**free Hamiltonian**) on the Fock space.

Proposition 2.1.28. ***a)*** *The subspace $\mathcal{D}_c := Span\{\mathcal{W}(\xi)\Omega\ ;\ \xi \in \mathcal{Z}\}$ is dense in the Fock space $\Gamma_s(\mathcal{Z})$.*
b) *For any $b \in \mathcal{P}_{alg}(\mathcal{Z})$ the operator b^{Wick} is closable and the domain of his closure contains*

$$\mathcal{H}_0 = Span\{\mathcal{W}(\phi)\psi, \psi \in \Gamma_s^{fin}(\mathcal{Z}), \phi \in \mathcal{Z}\}.$$

c) *For any $b \in \mathcal{P}$ the subspace $\mathcal{D}_c$ is a core for b^{Wick}.*
d) *For any $b(z) \in \bigoplus_{j=0}^r \mathcal{P}_{j,j}(\mathcal{Z})$ satisfies the following properties:*

$$b(z+i\pi\varepsilon\xi) = \sum_{j=0}^r \frac{(i\varepsilon\pi)^j}{j!}\mathrm{D}^j[b(z)][\xi],\ \xi \in \mathcal{Z},$$

where $\mathrm{D}^j[b(z)][\xi]$ the j-th differential of b with respect to $(z,\bar z)$ evaluated at ξ, i.e:

$$\mathrm{D}^j[b(z)][\xi] = \sum_{|\alpha|+|\beta|=j}\frac{j!}{\alpha!\beta!}\langle\xi^{\otimes\beta}\ ,\ \partial_z^\alpha\partial_{\bar z}^\beta b(z)\xi^{\otimes\alpha}\rangle.$$

Moreover there exists a ε-independent constant $C_r > 0$ such that

$$\|\langle\mathbf{N}\rangle^{-\frac{r}{2}}\sum_{j=0}^r\frac{(i\varepsilon\pi)^j}{j!}(\mathrm{D}^j[b(z)][\xi])^{Wick}\langle\mathbf{N}\rangle^{-\frac{r}{2}}\|_{\mathcal{L}(\Gamma_s(\mathcal{Z}))} \leqslant C_r\langle\xi\rangle^r.$$

e) *For all $\xi \in \mathcal{Z}$, the equality*

$$\mathcal{W}(\sqrt{2}\pi\xi)^* b^{Wick}\mathcal{W}(\sqrt{2}\pi\xi) = \{b(z+i\pi\varepsilon\xi)\}^{Wick}. \tag{2.1.11}$$

holds on $\mathcal{H}_0$.
f) *Let A be a self-adjoint operator on $\mathcal{Z}$ then for all $t \in \mathbb{R}$,*

$$e^{i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}d\Gamma(A)} = \big(b(e^{-itA}z)\big)^{Wick}. \tag{2.1.12}$$

2.1.4 Extension to Wick sesquilinear forms

The Wick quantization as we introduced in the previous section is a map that corresponds to a monomial $z \in \mathcal{Z} \mapsto b(z)$ an operator on the Fock space (the function $b(z)$ is called a symbol in connection with the pseudo-differential calculus). We extend below the class of symbols $\mathcal{P}_{p,q}(\mathcal{Z})$ to a wider one denoted $\mathcal{Q}_{p,q}(A)$, where A is a given non-negative self-adjoint operator on $\mathcal{Z}$. Consider for instance the mean field Hartree energy (see Section 5, Definition (5.2.7)),

$$h(z) = \langle z, Az\rangle + \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2})\,.$$

Then one observes that the symbol h is not in $\mathcal{P}_{alg}(\mathcal{Z})$ unless A and q are bounded. So, in order to extend the above quantization procedure to more interesting symbols, we introduce below the class $\mathcal{Q}_{p,q}(A)$.
Let A be a given non-negative self-adjoint operator on $\mathcal{Z}$. Let H_n^0 denotes, for each $n \in \mathbb{N}$, the operator on $\bigvee^n \mathcal{Z}$

$$H^0_{n_{|\bigvee^n \mathcal{Z}}} = \sum_{i=1}^n A_i\,.$$

For simplicity we denote

$$\mathfrak{Q}_n := Q(H_n^0) \subset \bigvee^n \mathcal{Z} \quad \text{and} \quad Q_n := Q(\sum_{i=1}^n A_i) \subset \otimes^n \mathcal{Z}\,,$$

with Q_n is a subspace possessing non symmetric vectors satisfying $\mathfrak{Q}_n \subset Q_n$, $\mathcal{S}_n Q_n = \mathfrak{Q}_n$ and $Q_n, \mathfrak{Q}_n$ are respectively dense in $\otimes^n \mathcal{Z}, \bigvee^n \mathcal{Z}$. Remember that Q_n and $\mathfrak{Q}_n$ are Hilbert spaces when they are equipped with the graph norm

$$\|u\|_{Q_n} = \|u\|_{\mathfrak{Q}_n} = \sqrt{\langle u, \sum_{i=1}^n A_i + 1\, u\rangle}\,, \quad \forall u \in Q_n. \tag{2.1.13}$$

We denote by Q_n' and $\mathfrak{Q}_n'$ respectively the dual spaces of Q_n and $\mathfrak{Q}_n$ with respect to the scalar product of $\otimes^n \mathcal{Z}_0$.
For all $p, q \in \mathbb{N}$, we define the class of symbols $\mathcal{Q}_{p,q}(A)$ as the space of complex-valued monomials on $Q(A)$ verifying

$$b \in \mathcal{Q}_{p,q}(A) \Leftrightarrow \exists!\, \tilde{b} \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}_q'),\ \forall z \in Q(A),\ b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p}\rangle_{\otimes^q \mathcal{Z}}\,. \tag{2.1.14}$$

Let $b \in \mathcal{Q}_{p,q}(A)$ and $\tilde{b}$ as in (2.1.14), then the map defined for any $\varphi_1, \cdots, \varphi_n \in Q(A)$ by

$$\tilde{b} \otimes 1^{(n-p)}\ \mathcal{S}_p \otimes 1^{(n-p)}\, \varphi_1 \otimes \cdots \otimes \varphi_n = \Big(\tilde{b}\, \mathcal{S}_p\, (\varphi_1 \otimes \cdots \varphi_p)\Big) \otimes \varphi_{p+1} \cdots \otimes \varphi_n\,, \tag{2.1.15}$$

extends by linearity and continuity to a bounded operator from Q_n into Q'_{n-p+q} since for any $\Phi^{(n)} \in \otimes^{alg,n} Q(A)$,

$$\begin{aligned}\|\tilde{b} \otimes 1^{(n-p)}\ \mathcal{S}_p \otimes 1^{(n-p)}\, \Phi^{(n)}\|_{Q'_{n-p+q}} &= \|(\sum_{i=1}^{n-p+q} A_i + 1)^{-\frac{1}{2}} \mathcal{S}_q \tilde{b} \mathcal{S}_p (\sum_{i=1}^{p} A_i + 1)^{-\frac{1}{2}} (\sum_{i=1}^{p} A_i + 1)^{\frac{1}{2}} \Phi^{(n)}\|_{\otimes^{n-p+q}\mathcal{Z}} \\ &\leqslant \|\tilde{b}\|_{\mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}_q')} \|\Phi^{(n)}\|_{Q_n}\,,\end{aligned}$$

and the subspace $\otimes^{alg,n} Q(A)$ is a form core for $\sum_{i=1}^{n} A_i$. As a consequence, we see that

$$\mathcal{S}_{n-p+q}\,\tilde{b}\otimes 1^{(n-p)}\,\mathcal{S}_n = \mathcal{S}_{n-p+q}\,\tilde{b}\otimes 1^{(n-p)}\,\mathcal{S}_p\otimes 1^{(n-p)}\,\mathcal{S}_n \in \mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})\,.$$

Definition 2.1.29. *For each symbol $b \in \mathcal{Q}_{p,q}(A)$, with $\tilde{b}$ as in* (2.1.14)*, we associate an operator b^{Wick}: $\bigoplus_{n\geqslant 0}^{alg} \mathfrak{Q}_n \longrightarrow \bigoplus_{n\geqslant 0}^{alg} \mathfrak{Q}'_n$, given by*

$$b^{Wick}_{|\mathfrak{Q}_n} = 1_{[p,+\infty)}(n)\frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\mathcal{S}_{n-p+q}(\tilde{b}\otimes 1^{\otimes(n-p)}) \in \mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})\,. \tag{2.1.16}$$

Actually $b^{Wick}_{|\mathfrak{Q}_n}$ can also be understood as a bounded sesquilinear form on $\mathfrak{Q}_n \times \mathfrak{Q}_{n-p+q}$. Remark that we have always the inclusion $\mathcal{P}_{p,q}(\mathcal{Z}) \subset \mathcal{Q}_{p,q}(A)$. Furthermore, the class $\mathcal{Q}_{p,q}(A)$ depends on the operator A and if A is bounded on $\mathcal{Z}$ then $\mathcal{Q}_{p,q}(A)$ coincides with $\mathcal{P}_{p,q}(\mathcal{Z})$.

Examples: Let q be a quadratic form on Q_2 that is a relatively form bounded with respect to A_1+A_2 with relative bound $\alpha < 2$ (see Section 5, Assumption (**A2**)). As a consequence of the above assumption, q can be identified with a bounded operator $\tilde{q}$ satisfying the relation:

$$q(u,v) = \langle u, \tilde{q}\, v\rangle_{\otimes^2\mathcal{Z}}\,, \quad \forall u,v \in Q(A_1+A_2)\,, \tag{2.1.17}$$

and $\tilde{q}$ acts from the Hilbert space $Q(A_1+A_2)$ equipped with the graph norm into its dual $Q'(A_1+A_2)$ with respect to the inner product of $\otimes^2\mathcal{Z}$. The main examples of interest here are

$$\begin{aligned} b_0(z) &= \langle z, Az\rangle \in \mathcal{Q}_{1,1}(A) && \text{with} \quad \tilde{b}_0 = A\,,\\ b(z) &= q(z^{\otimes 2}, z^{\otimes 2}) \in \mathcal{Q}_{2,2}(A) && \text{with} \quad \tilde{b} = \mathcal{S}_2\tilde{q}\mathcal{S}_2\,, \end{aligned}$$

and

$$h(z) = \langle z, Az\rangle + \frac{1}{2}q(z^{\otimes 2}, z^{\otimes 2}) \in \mathcal{Q}_{1,1}(A) + \mathcal{Q}_{2,2}(A)\,. \tag{2.1.18}$$

So using the Wick quantization given in Definition 2.1.29, one obtains the following equality in the sense of quadratic forms for any $\Psi^{(N)}, \Phi^{(N)} \in \mathfrak{Q}_N$,

$$\langle \Psi^{(N)}, H_N \Phi^{(N)}\rangle = \langle \Psi^{(N)}, \varepsilon^{-1} h^{Wick}\Phi^{(N)}\rangle, \quad \text{when} \quad \varepsilon = \frac{1}{N}\,,$$

where H_N is the N-body Hamiltonian defined in (1.1.2). This identity shows the relationship between the Hamiltonian of many-boson systems in the mean-field scaling and the Wick quantization of symbols in $\mathcal{Q}_{p,q}(A)$ with the semiclassical parameter ε. In fact most of the information we need in the analysis of the mean field approximation comes from general properties of the classes $\mathcal{Q}_{p,q}(A)$ stated in Proposition 2.1.30 below.

The linear space $\mathcal{Q}_{p,q}(A)$ is a subset of the space of continuous functions on $Q(A)$ and can be equipped with a convenient convergence topology. We say that a sequence $(c_m)_{m\in N}$ in $\mathcal{Q}_{p,q}(A)$ is b-convergent to a function $c(z)$ iff :

$$c_m \xrightarrow{b} c \Leftrightarrow \forall z \in Q(A),\ c_m(z) \to c(z) \text{ and } (||\tilde{c}_m||_{\mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_q)})_{m\in\mathbb{N}} \text{ is bounded }.$$

The following Proposition extends the previous Proposition 2.1.28 for symbols in $\mathcal{Q}_{p,q}(A)$.

Proposition 2.1.30. *For any $b \in \mathcal{Q}_{p,q}(A)$ and $(c_m)_{m\in N}$ a sequence in $\mathcal{Q}_{p,q}(A)$, we have:*
a) *$\bar{b} \in \mathcal{Q}_{q,p}(A)$ and*

$$\left(b^{Wick}_{|\mathfrak{Q}_n}\right)^* = \bar{b}^{Wick}_{|\mathfrak{Q}_{n-p+q}}\,.$$

b) *For any $t \in \mathbb{R}$, $b_t(z) := b(e^{-itA}z) \in \mathcal{Q}_{p,q}(A)$ with*

$$e^{i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}d\Gamma(A)} = b_t^{Wick}\,.$$

c) *There exists a constant $C_{p,q} > 0$ such that for any $\Psi^{(n)} \in \mathfrak{Q}_n$, $\Phi^{(m)} \in \mathfrak{Q}_m$ with $m = n-p+q$ and $\varepsilon = \frac{1}{n}$,*

$$\left|\langle \Phi^{(m)}, b^{Wick}\,\Psi^{(n)}\rangle\right| \leqslant C_{p,q}\,\left\|\tilde{b}\right\|_{\mathcal{L}(\mathfrak{Q}_n,\mathfrak{Q}'_m)}\,\left\|(A_1+1)^{\frac{1}{2}}\Phi^{(m)}\right\|\,\left\|(A_1+1)^{\frac{1}{2}}\Psi^{(n)}\right\|\,.$$

d) *If $c_m \xrightarrow{b} c$ then $c \in \mathcal{Q}_{p,q}(A)$ and c_m^{Wick} converges weakly to c^{Wick} in $\mathcal{L}(\mathfrak{Q}_n, \mathfrak{Q}'_{n-p+q})$.*
e) *For any $\xi \in Q(A)$ the symbol $b(\cdot+\xi)$ belongs to $\oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{Q}_{p,q}(A)$ and the identity*

$$b^{Wick}\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi) = \mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)b(z+\xi)^{Wick}\,, \tag{2.1.19}$$

holds in the sense of sesquilinear forms on $\mathfrak{Q}_{n_1}\times\mathfrak{Q}_{n_2}$ for any $n_1, n_2 \in \mathbb{N}$.

Proof. **a)** According to (2.1.14), we have

$$\bar{b}(z) = \overline{b(z)} = \langle \tilde{b}\,z^{\otimes q}, z^{\otimes p}\rangle = \langle z^{\otimes q}, \tilde{b}^*\,z^{\otimes p}\rangle\,,$$

where $\tilde{b}^* \in \mathcal{L}(\mathfrak{Q}_q, \mathfrak{Q}'_p)$ is the adjoint of $\tilde{b} \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)$. Let $\Phi^{(n)} \in \vee^{alg,n}Q(A)$, $\Psi^{(m)} \in \vee^{alg,m}Q(A)$ with $m = n-p+q$, $n \geqslant p$, then we have

$$\begin{aligned}
\langle b^{Wick}\Phi^{(n)}, \Psi^{(m)}\rangle &= \frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\,\langle \tilde{b}\otimes 1^{\otimes(n-p)}\,\Phi^{(n)}, \Psi^{(m)}\rangle \\
&= \frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\,\langle \Phi^{(n)}, \tilde{b}^*\otimes 1^{\otimes(n-p)}\Psi^{(m)}\rangle \\
&= \langle \Phi^{(n)}, \bar{b}^{Wick}\Psi^{(m)}\rangle\,.
\end{aligned}$$

Since $\vee^{alg,n}Q(A)$ is dense in the Hilbert space $(\mathfrak{Q}_n, ||.||_{\mathfrak{Q}_n})$, the above identity extends to any $\Phi^{(n)} \in \mathfrak{Q}_n$ and $\Psi^{(m)} \in \mathfrak{Q}_m$.
b) For any $t \in \mathbb{R}$ and $n \in \mathbb{N}$ the operator $(e^{-itA})^{\otimes n} : \mathfrak{Q}_n \to \mathfrak{Q}_n$ is bounded and extends by duality to a bounded operator on $\mathfrak{Q}'_n$. Hence for any $z \in Q(A)$,

$$b_t(z) := \langle (e^{-itA}z)^{\otimes q}, \tilde{b}\,(e^{-itA}z)^{\otimes p}\rangle = \langle z^{\otimes q}, (e^{-itA})^{\otimes q}\,\tilde{b}\,(e^{-itA})^{\otimes p}\,z^{\otimes p}\rangle\,,$$

and $\tilde{b}_t = (e^{-itA})^{\otimes q}\,\tilde{b}\,(e^{-itA})^{\otimes p}$ belongs to $\mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}'_q)$. Let $\Phi^{(n)} \in \mathfrak{Q}_n$, $\Psi^{(m)} \in \mathfrak{Q}_m$ with $m = n-p+q$, $n \geqslant p$, then we have

$$\begin{aligned}
\langle \Psi^{(m)}, e^{i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}d\Gamma(A)}\Phi^{(n)}\rangle &= \frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\,\langle \Psi^{(m)}, (e^{-itA})^{\otimes m}\,\tilde{b}\otimes\,(e^{-itA})^{\otimes n}\Phi^{(n)}\rangle \\
&= \frac{\sqrt{n!m!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\,\langle \Psi^{(m)}, \tilde{b}_t\otimes 1^{\otimes(n-p)}\Phi^{(n)}\rangle \\
&= \langle \Psi^{(m)}, b_t^{Wick}\Phi^{(n)}\rangle\,.
\end{aligned}$$

c) A simple estimate gives

$$\begin{aligned}
\left|\langle\Phi^{(m)}, b^{Wick}\,\Psi^{(n)}\rangle\right| &\leqslant \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\,\Big|\Big\langle (H_q^0+1)^{\frac12}\otimes 1^{\otimes(m-q)}\Phi^{(m)};\\
&\qquad \left((H_q^0+1)^{-\frac12}\tilde b(H_p^0+1)^{-\frac12}\right)\otimes 1^{\otimes(n-p)}\,(H_p^0+1)^{\frac12}\otimes 1^{\otimes(n-p)}\Psi^{(n)}\Big\rangle\Big|\\
&\leqslant \left\|\tilde b\right\|_{\mathcal{L}(\mathfrak{Q}_n,\mathfrak{Q}'_m)}\,\left\|(H_q^0+1)^{\frac12}\otimes 1^{\otimes(m-q)}\Phi^{(m)}\right\|\,\left\|(H_p^0+1)^{\frac12}\otimes 1^{\otimes(n-p)}\Psi^{(n)}\right\|.
\end{aligned}$$

Using the symmetry of the vectors $\Phi^{(m)}$ (resp. $\Psi^{(n)}$), we remark

$$\left\|(H_q^0+1)^{\frac12}\otimes 1^{\otimes(m-q)}\Phi^{(m)}\right\|^2 = \langle\Phi^{(m)}, (\sum_{i=1}^{q} A_i+1)\,\Phi^{(m)}\rangle = \langle\Phi^{(m)}, (qA_1+1)\,\Phi^{(m)}\rangle\,.$$

d) Thanks to a polarization formula the monomial c_m determines uniquely the operator $\tilde c_m \in \mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_q)$. In fact for any $\Phi^{(q)} \in \vee^{alg,q}Q(A)$ and $\Psi^{(p)} \in \vee^{alg,q}Q(A)$ the quantity $\langle\Phi^{(q)}, \tilde c_m\,\Psi^{(p)}\rangle$ can be written as a linear combination of $(c_m(z_i))_{i\in I}$ where I is a finite set and z_i are given points in $Q(A)$. Therefore, for any $\Phi^{(q)} \in \vee^{alg,q}Q(A)$ and $\Psi^{(p)} \in \vee^{alg,p}Q(A)$ the sequence $(\langle\Phi^{(q)}, \tilde c_m\,\Psi^{(p)}\rangle)_{m\in\mathbb{N}}$ is convergent. Since $(||\tilde c_m||_{\mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_q)})_{m\in\mathbb{N}}$ is bounded, one can prove by an $\eta/3$-argument that $\tilde c_m$ converges weakly to an operator $\tilde c \in \mathcal{L}(\mathfrak{Q}_p,\mathfrak{Q}'_q)$, i.e.:

$$\langle\Phi^{(q)}, \tilde c_m\,\Psi^{(p)}\rangle \underset{m\to\infty}{\to} \langle\Phi^{(q)}, \tilde c\,\Psi^{(p)}\rangle\,,\quad \forall\Phi^{(q)}\in\mathfrak{Q}_q, \forall\Psi^{(p)}\in\mathfrak{Q}_p\,. \tag{2.1.20}$$

Hence, $c(z) = \langle z^{\otimes q}, \tilde c z^{\otimes p}\rangle$ and belongs to $\mathcal{Q}_{p,q}(A)$. As a consequence of (2.1.20), the operator $\tilde c_m \otimes 1^{(n-p)}$ converges also weakly to $\tilde c\otimes 1^{(n-p)}$ in $\mathcal{L}(\mathfrak{Q}_n,\mathfrak{Q}'_{n-p+q})$ and the convergence of c_m^{Wick} towards c^{Wick} follows.

e) The relation (2.1.19) is already proved in [9, Proposition 2.10] for symbols $b \in \mathcal{P}_{p,q}(\mathcal{Z}_0)$. In order to extend it to the class $\mathcal{Q}_{p,q}(A)$ it is enough to use the approximation argument provided by (iv). Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ if $\|x\| \leqslant 1$, $\chi(x) = 0$ if $\|x\| \geqslant 2$ and $0 \leqslant \chi \leqslant 1$. We denote for $m\in\mathbb{N}$, $\chi_m(x) = \chi(\frac{x}{m})$. Let $b \in \mathcal{Q}_{p,q}(A)$ and consider the sequence of symbols

$$c_m(z) = \langle z^{\otimes q}, \chi_m(H_q^0)\,\tilde b\,\chi_m(H_p^0)\,z^{\otimes p}\rangle \in \mathcal{P}_{p,q}(\mathcal{Z}_0) \subset \mathcal{Q}_{p,q}(A)\,.$$

The use of [9, Proposition 2.10] yields for any $\Phi^{(n_1)} \in \mathfrak{Q}_{n_1}$ and $\Psi^{(n_2)} \in \mathfrak{Q}_{n_2}$,

$$\langle\Phi^{(n_1)}, c_m^{Wick}\mathcal{W}(\frac{\sqrt2}{i\varepsilon}\xi)\Psi^{(n_2)}\rangle = \langle\Phi^{(n_1)}, \mathcal{W}(\frac{\sqrt2}{i\varepsilon}\xi)\,c_m(z+\xi)^{Wick}\,\Psi^{(n_2)}\rangle\,. \tag{2.1.21}$$

Now, it is easy to check that

$$c_m \overset{b}{\to} b\,,\quad \text{in } \mathcal{Q}_{p,q}(A).$$

Moreover $c_m(\cdot+\xi) \in \oplus_{k,l\geqslant0}^{alg}\mathcal{P}_{k,l}(\mathcal{Z}_0)$,

$$\begin{aligned}
c_m(z+\xi) &= \langle (z+\xi)^{\otimes q}, \tilde c_m\,(z+\xi)^{\otimes p}\rangle\\
&= \sum_{\substack{0\leqslant i\leqslant q\\0\leqslant j\leqslant p}} C_q^i C_p^j\,\langle z^{\otimes(q-i)}\otimes\xi^{\otimes i}, \mathcal{S}_q\,\tilde c_m\,\mathcal{S}_p\,z^{\otimes(p-j)}\otimes\xi^{\otimes j}\rangle\\
&=: \sum_{\substack{0\leqslant i\leqslant q\\0\leqslant j\leqslant p}} C_q^i C_p^j\,c_m^{(i,j)}(z)\,.
\end{aligned}$$

So, it is clear that each monomial $c_m^{(i,j)}$ in the above sum b-converges to

$$b^{(i,j)} = \langle z^{\otimes(q-i)} \otimes \xi^{\otimes i}, \mathcal{S}_q\, \tilde{b}\, \mathcal{S}_p\, z^{\otimes(p-j)} \otimes \xi^{\otimes j} \rangle,$$

since $\tilde{c}_m$ converges weakly to $\tilde{b}$ in $\mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}_q')$. Remark also that Proposition 2.1.31 shows for any $r \in \mathbb{N}$ that the r^{th} components of the following coherent vectors satisfy

$$\big[\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)\Psi^{(n_2)}\big]^{(r)} \in \mathfrak{Q}_r \quad \text{and} \quad \big[\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}\xi)^*\Phi^{(n_1)}\big]^{(r)} \in \mathfrak{Q}_r\,.$$

Therefore using (iv) and taking the limit $m \to \infty$ in (2.1.21) proves the claimed identity. ∎

A regularity property of Weyl operators: It is convenient to recall the following regularity property for the Weyl operators. Remember that the operator $d\Gamma(A) + \mathbf{N}$ is non-negative and self-adjoint on the symmetric Fock space satisfying

$$d\Gamma(A) + \mathbf{N}_{|\vee^N \mathcal{Z}_0} = \frac{H_N^0}{N} + 1\,, \qquad \text{when } \varepsilon = \frac{1}{N}\,.$$

Moreover, $d\Gamma(A) + \mathbf{N}$ has an invariant form domain with respect to the Weyl operator $\mathcal{W}(\xi)$ when $\xi \in Q(A)$. This propriety can be proved using the Faris-Lavine argument [45] and it is proved for instance in [6].

Proposition 2.1.31. *For any $\xi \in Q(A)$ the form domain $Q(d\Gamma(A) + \mathbf{N})$ is invariant with respect to the Weyl operator $\mathcal{W}(\xi)$. Moreover, there exists uniformly in $\varepsilon \in (0, \bar{\varepsilon})$ a constant $C := C(\xi) > 0$ such that*

$$\|(d\Gamma(A) + \mathbf{N})^{\frac{1}{2}} \mathcal{W}(\xi)(d\Gamma(A) + \mathbf{N} + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(\Gamma_s(\mathcal{Z}_0))} \leqslant C\,, \tag{2.1.22}$$

and in particular for any $\Psi^{(N)} \in Q(H_N^0)$, $\varepsilon = \frac{1}{N}$,

$$\left\|(\frac{H_{N-1}^0}{N-1} + 1)^{1/2}[\mathcal{W}(\xi)\Psi^{(N)}]^{(N-1)}\right\| \leqslant C \left\|(\frac{H_N^0}{N} + 1)^{1/2}\Psi^{(N)}\right\|,$$

where $[\mathcal{W}(\xi)\Psi^{(N)}]^{(N-1)}$ denotes the $(N-1)^{th}$ component of $\mathcal{W}(\xi)\Psi^{(N)} \in \Gamma_s(\mathcal{Z}_0)$.

2.2 Wigner measures

Semi-classical (or Wigner) measures are well-known tools in non-linear analysis in finite dimension. We refer the reader to the work [9, 27, 52, 51, 62, 113] for examples. The Wigner measures provide a powerful tool to get the leading term in the semiclassical limit. Its strong link with the phase-space analysis allows the derivation of some a priori estimates that are useful in the study of the mean field limit as we will see later. In particular, these measures make the link between the pseudodifferential calculus and the phase-space geometry. In the first subsection we will also introduce the well-known notion of normal states in the Fock space. In the second 2.2.2, we will extend the notion of Wigner measures in a infinite dimensional Hilbert space by a projective approach according to the work [9]. The tensor decomposition is a useful property of Fock spaces and several information on Wigner measures can be deduced from this fact. Several equivalent definitions for the Wigner measures under some uniform trace conditions exist, but we should be very careful with the choice of the quantized observables. Indeed there is a lack of compactness owing to the infinite dimensional Hilbert space considered and then the Wigner convergence is harder to get in this framework. Subsequently, we shall consider Wick observables with compact kernels to prevent this problem. This is explained in subsection 2.2.3 where we will introduce the reduced density matrices which are also useful to understand the mean field limit (see [42, 43, 11, 18, 110, 47, 49]). We mentioned earlier that Wigner measures can be used to derive some a priori estimates. In [12, 78, 7] these estimates have been used to make a link between the quantum dynamics and the mean field dynamics. In the last subsection 2.2.4 we will emphasize the link between Wick symbols and the Wigner measures throughout some a priori estimates.

2.2.1 Semi-classical measures in finite dimension

Consider a finite dimensional space $\mathcal{H}$. In the following $\mathcal{E}$ denotes an infinite subset of $(0,+\infty)$ such that $0 \in \overline{\mathcal{E}}$. Recall that the space $\mathcal{L}^1(\mathcal{H})$ denotes the space of trace-class operators on $\mathcal{H}$.

Definition 2.2.1. *The family* $(\varrho_h)_{h\in\mathcal{E}} \in \mathcal{L}^1(\mathcal{H})$ *is a family of normal states if and only if*

$$\begin{aligned} &\mathbf{i})\ \varrho_h \geqslant 0 \\ &\mathbf{ii})\ \mathrm{Tr}\,[\varrho_h] = 1. \end{aligned}$$

Consider $(\varrho_h)_{h\in\mathcal{E}}$ a family of normal states on $\mathbb{R}^{2d}$. Let $\mathcal{M}_b(\mathbb{R}^{2d})$ be the set of bounded Radon measures on $\mathbb{R}^{2d}$. Notice that $\mathcal{M}_b(\mathbb{R}^{2d})$ is the dual of the separable space of continuous functions with limit 0 at infinity, namely $\mathcal{C}_0(\mathbb{R}^{2d})$. Hence for $R > 0$, the ball

$$B_R = \{\mu \in \mathcal{M}_b(\mathbb{R}^{2d}),\ \|\mu\|_{\mathcal{M}_b(\mathbb{R}^{2d})} \leqslant R\},$$

with

$$\|\mu\|_{\mathcal{M}_b(\mathbb{R}^{2d})} = \sup_{f\in\mathcal{C}_0(\mathbb{R}^{2d})} \frac{|\int_{\mathbb{R}^{2d}} f(x)d\mu(x)|}{\|f\|_{\mathcal{C}_0(\mathbb{R}^{2d})}},$$

is metrizable. Therefore bounded subsets are relatively sequentially compact for the weak-* topology.

Definition 2.2.2. *For the family of normal states* $(\varrho_h)_{h\in\mathcal{E}}$*, the semi-classical measure (or Wigner measure) are the weak-* limit points of* $\frac{1}{(2\pi h)^d}\sigma^{Wick}(\varrho_h)$ *in* $\mathcal{M}_b(\mathbb{R}^d)$.
The set of semi-classical measures associated with $(\varrho_h)_{h\in\mathcal{E}}$ *is denoted by* $\mathcal{M}(\varrho_h,\ h\in\mathcal{E})$. *Moreover the family* $(\varrho_h)_{h\in\mathcal{E}}$ *is said pure if* $\mathcal{M}(\varrho_h,\ h\in\mathcal{E})$ *is reduced to a single element.*

Remarks 2.2.3. *The same definitions can be used for general bounded family* $(\varrho_h)_{h\in\mathcal{E}}$ *in* $\mathcal{L}(L^2(\mathbb{R}^d))$. *The extension of this definition to a separable Hilbert space will be discuss later in Section 2.2.2.*

Proposition 2.2.4. i) *When* $(\varrho_h)_{h\in\mathcal{E}}$ *is a family of normal states, any* $\mu \in \mathcal{M}(\varrho_h,\ h\in\mathcal{E})$ *satisfies*

$$0 \leqslant \mu\ ,\ \int_{\mathbb{R}^d} d\mu \leqslant 1.$$

(ii) *The Wigner measure $\mu \in \mathcal{M}(\varrho_h,\ h \in \mathcal{E})$ is characterized by the assertion:*
There exists a sequence $(h_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} h_n = 0$ such that for any $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$

$$\lim_{n\to+\infty} \mathrm{Tr}\,[b^{\#}(x, h_n D_x)\varrho_{h_n}] = \int_{\mathbb{R}^{2d}} b(X)d\mu(X),$$

where $\#$ stands for Weyl or Anti-Wick quantization.
iii) *If there is no loss of mass at infinity, i.e: $\mathrm{Tr}\,[\varrho_h \mathbf{N}^\nu] \leqslant C_\nu < \infty$ uniformly w.r.t $h \in \mathcal{E}$, and for some $\nu > 0$ with $\mathbf{N}$ the harmonic oscillator introduced in* (**Number operator**)*, then any $\mu \in \mathcal{M}(\varrho_h,\ h \in \mathcal{E})$ is a probability measure. With this assumption, the Wigner measure μ is also characterized by the equality*

$$\lim_{n\to\infty} \mathrm{Tr}\,[\mathcal{W}(\frac{1}{i\sqrt{2}}X_0)\varrho_h] = \int_{\mathbb{R}^{2d}} e^{i\sigma(X_0,X)}\ d\mu(X),$$

with σ denotes the symplectic form on $\mathbb{R}^{2d}$.

Examples 2.2.5.

1. *Consider the family $(\varrho_h = |\varphi_{X_0}\rangle\langle\varphi_{X_0}|)_{h\in(0,h_0)}$, with $h_0 > 0$. Then the Wigner measure associated with ϱ_h is the Dirac measure centered in X_0, i.e:*

$$\mathcal{M}(\varrho_h,\ h \in (0, h_0)) = \{\delta_{X_0}\}.$$

2. *Any probability measure on $\mathbb{R}^{2d}$ is a semi-classical measure of a given family of normal states.*

3. *When $d = 1$, and $\varrho_h = |\Psi_\alpha\rangle\langle\Psi_\alpha|$ with $\alpha = [\frac{1}{h}]$, then*

$$\mathcal{M}(\varrho_h, h \in (0, h_0)) = \{\frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}}\ d\theta\} = \{\delta_1^{S^1}\}.$$

2.2.2 Wigner measures in a infinite dimensional Hilbert space

First definitions

The extension proposed here follows the one in [12] and purposes a projective approach. In order to use the Bochner's Theorem 2.2.8, we consider real Hilbert spaces. In practice the real structure of a complex Hilbert spaces is considered. We begin by introducing the space $\mathbb{P}$ of all orthogonal projections of finite ranks on $\mathcal{Z}$. Define the usual Schwarz space in a finite dimensional space E by $\mathcal{S}(E)$.

Definition 2.2.6. *A function $f : \mathcal{Z} \to \mathbb{C}$ is said Schwartz cylindrical if there exists $\mathfrak{p} \in \mathbb{P}$, and a function $g \in \mathcal{S}(\mathfrak{p}\mathcal{Z})$, such that for any $z \in \mathcal{Z},\ f(z) = g(\mathfrak{p}z)$. In this case we said that f is based on $\mathfrak{p}\mathcal{Z}$. We denote $\mathcal{S}_{cyl}(\mathcal{Z})$ the cylindrical Schwarz space*

$$f \in \mathcal{S}_{cyl}(\mathcal{Z}) \Leftrightarrow \exists \mathfrak{p} \in \mathbb{P}, \exists g \in \mathcal{S}(\mathfrak{p}\mathcal{Z}),\ f(z) = g(\mathfrak{p}z).$$

Definition 2.2.7. *The algebra of cylindrical sets $\mathcal{B}_{cyl}(\mathcal{Z}) = \{X(\mathfrak{p}, E) = \mathfrak{p}^{-1}(E),\ \mathfrak{p} \in E, E \in \mathcal{B}(\mathfrak{p}\mathcal{Z})\}$ where $\mathcal{B}(\mathfrak{p}\mathcal{Z})$ denotes for any $\mathfrak{p} \in \mathbb{P}$ the set of Borel subsets of $\mathfrak{p}\mathcal{Z}$. A cylindrical measure μ is a mapping defined on $\mathcal{B}_{cyl}(\mathcal{Z})$ such that*

- **i)** $\mu(\mathcal{Z}) = 1$.
- **ii)** $\forall \mathfrak{p} \in \mathbb{P},\ \mu_{\mathfrak{p}}(A) := \mu(\mathfrak{p}^{-1}(A))$ *for $A \in \mathcal{B}_{cyl}(\mathcal{Z})$ defines a probability measure $\mu_{\mathfrak{p}}$ on $\mathcal{B}(\mathfrak{p}\mathcal{Z})$.*

In this context, the family of measures $(\mu_{\mathfrak{p}})_{\mathfrak{p}\in\mathbb{P}}$ is called weak distribution. The space $\mathcal{Z}$ equipped with its real scalar product is a real Hilbert space and the function $z \mapsto e^{2i\pi\,\mathrm{Re}\,\langle z,\xi\rangle_{\mathcal{Z}}}$, with $\xi \in \mathcal{Z}$ is a cylindrical measurable function. Hence we can define the Fourier transform of a cylindrical mesure μ

$$\mathcal{F}[\mu](\xi) = \int_{\mathcal{Z}} e^{2i\pi\,\mathrm{Re}\,\langle z,\xi\rangle_{\mathcal{Z}}}\, d\mu(z).$$

Below we recall the Bochner's theorem

Theorem 2.2.8. *A function G is a Fourier transform of a weak distribution if and only if*

i) $G(0) = 1$.

ii) *G is of positive type, i.e.,* $\sum_{i,j}^{N} \lambda_i \bar{\lambda}_j G(\xi_i - \xi_j) \geqslant 0$, *for any* $N \in \mathbb{N}$, $(\lambda_i)_{i=1,..,N} \in \mathbb{C}$, $(\xi_i)_{i=1,..,N} \in \mathcal{Z}$. (2.2.1)

iii) *For any* $\mathfrak{p} \in \mathbb{P}, G_{|\mathfrak{p}\mathcal{Z}}$ *is continuous.*

Before introducing the notion of Wigner measures, we recall a tightness property which can be found for instance in [107].

Lemma 2.2.9. *A cylindrical measure μ on $\mathcal{Z}$ extends to a probability measure on $\mathcal{Z}$ if and only if for any $\eta > 0$, there exists $R_\eta > 0$ such that*

$$\forall \mathfrak{p} \in \mathbb{P},\ \mu(\{z \in \mathcal{Z},\ \|\mathfrak{p}z\| \leqslant R_\eta\}) \geqslant 1 - \eta.$$

Definition 2.2.10. *Let $\mathcal{E}$ be an infinite subset of $(0, +\infty)$ such that $0 \in \overline{\mathcal{E}}$. Let $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$ ($\varrho_\varepsilon \geqslant 0$ and $\mathrm{Tr}\ [\varrho_\varepsilon] = 1$) such that:*

$$\exists\delta > 0\,, \exists C_\delta > 0\,, \forall\varepsilon \in \mathcal{E}\,, \qquad \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta < \infty\,.$$

The set $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$ of Wigner measures associated with $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ is the set of Borel probability measures on $\mathcal{Z}$, μ, such that there exists an infinite subset $\mathcal{E}' \subset \mathcal{E}$ with $0 \in \overline{\mathcal{E}'}$ and

$$\forall\xi \in \mathcal{Z}\,, \quad \lim_{\mathcal{E}'\ni\varepsilon\to 0} \mathrm{Tr}\ \left[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\right] = \int_{\mathcal{Z}} e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle} d\mu(z)\,.$$

Moreover the measure is satisfying

$$\int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^{2\delta}\, d\mu(z) < C_\delta < \infty.$$

Remarks 2.2.11. *The expression $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}$ means that the family $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ is pure in the sense:*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,[\varrho_\varepsilon\, b^{Weyl}] = \int_{\mathcal{Z}} b(z)\, d\mu(z),$$

for all cylindrical symbols $b \in \mathcal{S}_{cyl}(\mathcal{Z})$ without extracting a subsequence.

By some diagonal extraction of subsequences, it was proved in [9] that $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$ is never empty. Additionnally we can assume without loss of generality that $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}$. In practice the Wigner measures are identified though their characteristic functions with the relation:

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\} \Leftrightarrow \lim_{\varepsilon \to 0} \operatorname{Tr}\left[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\right] = \mathcal{F}^{-1}[\mu](\xi).$$

Remarks 2.2.12. *The definition of Wigner measures extends to every family $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ such that*

$$\operatorname{Tr}\left[(1+\mathbf{N})^\delta \varrho_\varepsilon (1+\mathbf{N})^\delta\right] \leqslant C_\delta,$$

for a fixed $\delta > 0$ and according to the decomposition

$$\varrho_\varepsilon = \lambda_\varepsilon^{R,+}\varrho_\varepsilon^{R,+} - \lambda_\varepsilon^{R,-}\varrho_\varepsilon^{R,-} + i\lambda_\varepsilon^{I,+}\varrho_\varepsilon^{I,+} - i\lambda_\varepsilon^{I,-}\varrho_\varepsilon^{I,-},$$

where $(\varrho_\varepsilon^{R/I,\pm})_{\varepsilon\in\mathcal{E}}$ are normal states.

Examples 2.2.13. *1. Consider the state $\varrho_\varepsilon = |\Omega\rangle\langle\Omega|$ for all $\varepsilon \in (0, \bar{\varepsilon})$, then*

$$\mathcal{M}(\varrho_\varepsilon,\, \varepsilon \in (0,\bar{\varepsilon}) = \{\delta_0\}.$$

2. When $\varrho_\varepsilon = \varrho_{\varepsilon,1} \otimes \varrho_{\varepsilon,2}$ in the tensor decomposition $\Gamma_s(\mathcal{Z}_1) \oplus \Gamma_s(\mathcal{Z}_2)$ with the uniform estimates

$$\operatorname{Tr}\left[\varrho_{\varepsilon,j}\mathbf{N}_j^\delta\right] \leqslant C_\delta,$$

$$\mathcal{M}(\varrho_{\varepsilon,j},\, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu_j\},\ \textit{for } j = 1, 2.$$

Then $\mathcal{M}(\varrho_\varepsilon,\, \varepsilon \in (0,\bar{\varepsilon}) = \{\mu_1 \otimes \mu_2\}$.

3. For the family of coherent states $\varrho_\varepsilon := |E(z)\rangle\langle E(z)|$, the Wigner measure associated is δ_z.

4. As in finite dimension, any Borel probability measure on $\mathcal{Z}$ is a Wigner measure.

Weyl and Anti-Wick Observables

Weyl operators: Denote $L_\mathfrak{p}(dz)$ the Lebesgue measure on the finite space $\mathfrak{p}\mathcal{Z}$, where $\mathfrak{p} \in \mathbb{P}$. We can define for a function $f \in \mathcal{S}_{cyl}(\mathcal{Z})$ its Fourier transform

$$\mathcal{F}[f](\xi) = \int_{\mathfrak{p}\mathcal{Z}} f(\xi) e^{-2i\pi \operatorname{Re}\langle z,\xi\rangle}\, L_\mathfrak{p}(d\xi).$$

Take any symbol $b \in \mathcal{S}_{cyl}(\mathcal{Z})$, a Weyl observable can be associated according to

$$b^{Weyl} = \int_{\mathfrak{p}\mathcal{Z}} \mathcal{F}[b](z)\mathcal{W}(\sqrt{2}\pi z) L_\mathfrak{p}(dz). \tag{2.2.2}$$

By using the tensor decomposition $\Gamma_s(\mathcal{Z}) = \Gamma_s(\mathfrak{p}\mathcal{Z}) \otimes \Gamma_s((1-\mathfrak{p})\mathcal{Z})$ owing to
$\mathcal{Z} = \mathfrak{p}\mathcal{Z} \overset{\perp}{\oplus} (1-\mathfrak{p})\mathcal{Z},\ \forall z \in \mathfrak{p}\mathcal{Z}$,

$$\mathcal{W}(\sqrt{2}\pi z) = \mathcal{W}_{\mathfrak{p}\mathcal{Z}}(\sqrt{2}\pi z) \otimes \text{ Id }_{|\Gamma_s((1-\mathfrak{p})\mathcal{Z})},$$

where $\mathcal{W}_{\mathfrak{p}\mathcal{Z}}(\sqrt{2}\pi z)$ denotes the reduced representation in $\Gamma_s(\mathfrak{p}\mathcal{Z})$. Then this projective aspect allows more general classes of symbols. Hence for $\mathfrak{p} \in \mathbb{P}$, the symbol classes defined for $0 \leqslant \nu \leqslant 1$ on $\mathfrak{p}\mathcal{Z}$,

$$S^{\nu}_{\mathfrak{p}\mathcal{Z}} = \bigoplus_{n\in\mathbb{Z}}^{alg} S(\langle z\rangle^{n}_{\mathfrak{p}\mathcal{Z}}, \frac{dz^2}{\langle z\rangle^{2\nu}_{\mathfrak{p}\mathcal{Z}}}),$$

where $\langle z\rangle^2_{\mathfrak{p}\mathcal{Z}} = 1 + \|z\|^2_{\mathfrak{p}\mathcal{Z}}$ is the japonese bracket. Polynomial functions on $\mathfrak{p}\mathcal{Z}$ are included in $S^{\nu}_{\mathfrak{p}\mathcal{Z}}$. The associated class of Weyl quantized operators after tenzorisation with $\text{ Id }_{\Gamma_s((1-\mathfrak{p})\mathcal{Z})}$ is denoted by Op $S^{\nu}_{\mathfrak{p}\mathcal{Z}}$. Take a cylindrical symbol $b \in \mathcal{P}_{alg}(\mathcal{Z})$ based on $\mathfrak{p}\mathcal{Z}$, the equivalence of the Weyl and Wick quantization in finite dimension gives the following equality in Op $S^{\nu}_{\mathfrak{p}\mathcal{Z}}$ for any $\nu \in [0,1]$

$$b^{Wick} = b^{Weyl} + O_b(\varepsilon).$$

However, these polynomials have finite range and make a dense set in $\mathcal{P}^{\infty}_{alg}(\mathcal{Z})$ but not in $\mathcal{P}_{alg}(\mathcal{Z})$ owing to a phenomen called infinite dimensional compactness defect. We talk about this issue in Section 2.2.3.

Anti-Wick operators:

We can define Anti-Wick operators by a separation of variables. Given $\mathfrak{p} \in \mathbb{P}$. Set $\mathfrak{p}^{\perp} = 1 - \mathfrak{p}$, and use the tensor decomposition (2.1.8). Then for $\xi_1 \overset{\perp}{\oplus} \xi_2$, $\mathcal{W}(\xi_1 \overset{\perp}{\oplus} \xi_2) = \mathcal{W}_{\mathfrak{p}}(\xi_1) \otimes \mathcal{W}_{\mathfrak{p}^{\perp}}(\xi_2)$. In finite dimension the coherent states $E_{\mathfrak{p}}(\xi) = \mathcal{W}_{\mathfrak{p}}(\frac{\sqrt{2}\xi}{i\varepsilon})\Omega_{\mathfrak{p}\mathcal{Z}}$. Introduce the projector P_{ξ} on $\Gamma_s(\mathcal{Z})$ after tenzorisation with $\text{ Id }_{\Gamma_s(\mathfrak{p}^{\perp}\mathcal{Z})}$:

$$\mathfrak{p}\mathcal{Z} \ni \xi \mapsto P^{\varepsilon}_{\xi} = (|E_{\mathfrak{p}}(\xi)\rangle\langle E_{\mathfrak{p}}(\xi)|) \otimes \text{ Id }_{|\Gamma_s(\mathfrak{p}^{\perp}\mathcal{Z})}.$$

For a symbol $b \in \mathcal{S}_{cyl}(\mathcal{Z})$ based on $\mathfrak{p}\mathcal{Z}$, the Anti-Wick operator is defined by

$$b^{A-Wick} = \int_{\mathfrak{p}\mathcal{Z}} b(\xi) P^{\varepsilon}_{\xi} \frac{L_{\mathfrak{p}}(d\xi)}{(\pi\xi)^{\dim\mathfrak{p}\mathcal{Z}}}.$$

The coherent states $|E_{\mathfrak{p}}(\xi)\rangle\langle E_{\mathfrak{p}}(\xi)|$ can be see as Weyl observables in finite dimension. The next table explains the correspondence

$\mathfrak{p}\mathcal{Z} \sim \mathbb{C}^d$	$z = x + iy\ ,\ \ T^*\mathbb{R}^d$
$\Gamma_s(\mathfrak{p}\mathcal{Z}) \sim \Gamma_s(\mathfrak{p}\mathcal{Z})$	$\varepsilon = 2h\ ,\ \ L^2(\mathbb{R}^d)$
$E_{\mathfrak{p}\mathcal{Z}}(z_0)$	$\frac{z_0}{i} = \xi_0 - ix_0\ ,\ \ \tau_{(\frac{x_0}{\sqrt{h}},\frac{\xi_0}{\sqrt{h}})}(\frac{1}{\pi^{\frac{d}{4}}} e^{-\frac{x^2}{2}})$
$\lvert\Omega^{\mathfrak{p}\mathcal{Z}}\rangle\langle\Omega^{\mathfrak{p}\mathcal{Z}}\rvert = \gamma^{Weyl},\ \gamma(z) = 2^d e^{-\frac{\lvert z\rvert^2_{\mathfrak{p}\mathcal{Z}}}{\frac{\varepsilon}{2}}}$	$\frac{1}{\pi^{\frac{d}{4}}} e^{-\frac{(x^2+y^2)}{2}} = g^{Weyl}(\sqrt{h}x, \sqrt{h}D_x),\ g(x,\xi) = 2^d e^{-\frac{x^2+\xi^2}{h}}.$

Hence by using the conjugation formula for $a^{Weyl}(\sqrt{h}x, \sqrt{h}D_x)$ w.r.t $\tau_{(\frac{x_0}{\sqrt{h}},\frac{\xi}{\sqrt{h}})}$ we get the correspondence

$$|E_{\mathfrak{p}}(\xi)\rangle\langle E_{\mathfrak{p}}(\xi)| = \gamma^{Weyl}_{\xi},\ \gamma_{\xi}(z) = 2^d e^{-\frac{|z-\xi|^2_{\mathfrak{p}\mathcal{Z}}}{\frac{\varepsilon}{2}}}.$$

Proposition 2.2.14. *1. As in finite dimension, the positivity property of $A - Wick$ operators holds for symbols in $S_{\mathfrak{p}\mathcal{Z}}(1, |dz|^2)$, for $\mathfrak{p} \in \mathbb{P}$.*

2. Let $\mathfrak{p} \in \mathbb{P}$, and $b \in S_{\mathfrak{p}\mathcal{Z}}(1, |dz|^2)$, then

$$|b^{A-Wick}|_{\mathcal{L}(\Gamma_s(\mathcal{Z}))} \leqslant \|b\|_{L^\infty(\mathfrak{p}\mathcal{Z})}.$$

3. The comparaison with the Weyl quantization for symbols in $S_{\mathfrak{p}\mathcal{Z}}(1, |dz|^2)$, for $\mathfrak{p} \in \mathbb{P}$ gives the estimate

$$\|b^{A-Wick} - b^{Weyl}\|_{\mathcal{L}(\Gamma_s(\mathcal{Z}))} \leqslant C_d p_{k_d}(b)\varepsilon,$$

where the constant $C_d > 0$ and the semi norm p_{k_d} depend on the dimension $d = dim\mathfrak{p}\mathcal{Z}$.

2.2.3 Lack of compactness

Recall that $\mathcal{E}$ denotes an infinite subset of $(0, +\infty)$ such that $0 \in \overline{\mathcal{E}}$. In the subsection we always consider a family of states $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ such that

$$\forall k \in \mathbb{N},\ \exists C_k > 0,\ \forall \varepsilon \in \mathcal{E},\ \ \mathrm{Tr}\ [\mathbf{N}^k \varrho_\varepsilon] \leqslant C_k,$$

and

$$\mathcal{M}(\varrho_\varepsilon,\, \varepsilon \in \mathcal{E}) = \{\mu\}.$$

Within these assumptions, the convergence

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\ [\varrho_\varepsilon b^{Weyl}] = \int_{\mathcal{Z}} b(z)\, d\mu(z), \tag{2.2.3}$$

holds for any cylindrical $b(z) = g(\mathfrak{p}z)$, with $g \in S(\langle z\rangle^n_{\mathfrak{p}\mathcal{Z}},\ \frac{dz^2}{\langle z\rangle^2_{\mathfrak{p}\mathcal{Z}}})$. In particular the convergence (2.2.3) holds for a polynomial g on $\mathfrak{p}\mathcal{Z}$, $\mathfrak{p} \in \mathbb{P}$. In finite dimension, the Wick quantization and the Weyl quantization are equivalent, therefore b^{Weyl} can be replaced by b^{Wick}. Moreover, a symbol $b(z) = \langle z^{\otimes q}, \tilde{b}z^{\otimes p}\rangle$ is cylindrical when $\tilde{b} \in \mathcal{L}(\bigvee^p \mathcal{Z}; \bigvee^q \mathcal{Z})$ has a finite rank. The convergence (2.2.3) holds for any $\tilde{b} \in \mathcal{L}^\infty(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$, see for example [10, Corollary 6.14].

Lemma 2.2.15. *Let $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$ depending on ε such that*

$$\forall \alpha \in \mathbb{N}, \exists C_\alpha > 0, \forall \varepsilon \in \mathcal{E}, \quad \mathrm{Tr}\ [\varrho_\varepsilon \mathbf{N}^\alpha] \leqslant C_\alpha, \text{ and } \mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}.$$

Then, there exists a infinite subset $\mathcal{E}' \subset \mathcal{E}$ with $0 \in \bar{\mathcal{E}}'$ such that

$$\lim_{\mathcal{E}'\ni\varepsilon\to 0} \mathrm{Tr}\ [\varrho_\varepsilon b^{Wick}] = \int_{\mathcal{Z}} b(z)\ d\mu(z), \tag{2.2.4}$$

for any $b \in \mathcal{P}^\infty_{alg}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}} \mathcal{P}^\infty_{p,q}(\mathcal{Z})$.

It is not possible to extend the convergence (2.2.4) for bounded polynomials $b \in \mathcal{P}$. The phenomena corresponds to the infinite dimensional defect of compactness (see [9, 78] for more details). There are several examples where the convergence (2.2.4) does not hold:

Examples 2.2.16. *1. Consider Coherent states $E(f_\varepsilon)$ where the vector f is ε dependant. Assume that the family $(f_\varepsilon)_{\varepsilon>0}$ is weakly convergent to f_0 but not strongly, with $\|f_\varepsilon\|_{\mathcal{Z}} = 1$. Then the family $\varrho_\varepsilon = |E(f_\varepsilon)\rangle\langle E(f_\varepsilon)|$ admit a single Wigner measure δ_{f_0}. However, take $b(z) = \|z\|^2_{\mathcal{Z}}$ and compute*

$$\mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}] = \|f_\varepsilon\|_{\mathcal{Z}} = 1 \neq \int_{\mathcal{Z}} \|z\|^2_{\mathcal{Z}}\, d\delta_{f_0}(z).$$

2. Consider Hermite states $\varrho_\varepsilon = |f_\varepsilon^{[\frac{1}{\varepsilon}]}\rangle\langle f_\varepsilon^{[\frac{1}{\varepsilon}]}|$, where the family of normal states $(f_\varepsilon)_\varepsilon$ is weakly convergent to f_0 but not strongly. A simple computation yields

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^k] = 1 \neq \int_{\mathcal{Z}} \|z\|^{2k}_{\mathcal{Z}}\, d\delta_{f_0}.$$

BBGKY Hierarchy:

In this subsection we assume that $\mathcal{Z} = L^2(\mathbb{R}^d)$. States will be considered in $\bigvee^n \mathcal{Z} = L^2(\mathbb{R}^{dn})$. Thus take $\varrho_\varepsilon \in \mathcal{L}(\bigvee^n \mathcal{Z})$, with $n = [\frac{1}{\varepsilon}]$. For any $p \in \mathbb{N},\ p \leqslant n, \gamma_\varepsilon^{(p)} \in \mathcal{L}^1(\bigvee^p \mathcal{Z})$ is defined as the partially traced operator with the kernel

$$\gamma_\varepsilon^{(p)}(x_1, ..., x_p; y_1, ..., y_p) := \int_{\mathbb{R}^{d(n-p)}} \varrho_\varepsilon(x_1, ..., x_p, X, y_1, ..., y_p, X)\ L_{\mathbb{R}^{d(n-p)}}(dX).$$

Proposition 2.2.17. *Assume that $\varrho_\varepsilon \in \mathcal{L}(\Gamma_s(\mathcal{Z}))$ satisfies $\varrho_\varepsilon \geqslant 0$ and $\mathrm{Tr}\,[\mathbf{N}^{\frac{k}{2}}\varrho_\varepsilon \mathbf{N}^{\frac{k}{2}}] < \infty$ for all $k \in \mathbb{N}$. Then for any $p \in \mathbb{N}$ the writing*

$$\mathrm{Tr}\,[\gamma_\varepsilon^{(p)}\tilde{b}\ ] = \mathrm{Tr}\,[\varrho_\varepsilon]\frac{\mathrm{Tr}\,[\varrho_\varepsilon b^{Wick}]}{\mathrm{Tr}\,[\varrho_\varepsilon(|z|^{2p})^{Wick}]}$$

for any $b \in \mathcal{P}_{p,p}(\mathcal{Z})$, defines a unique element $\gamma_\varepsilon^{(p)} \geqslant 0$ in $\mathcal{L}(\bigvee^p \mathcal{Z})$.

Proof. Owing to the assumptions the application $\tilde{b} \mapsto \mathrm{Tr}\,[\varrho_\varepsilon b^{Wick}]$ defines a continuous linear form on $\mathcal{L}(\bigvee^p \mathcal{Z})$. ∎

Hence the definition of reduced density matrix follows by this duality argument.

Definition 2.2.18. *Assume that the family $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ satisfy $\varrho_\varepsilon \geqslant 0$ and $\mathrm{Tr}\,[\mathbf{N}^{\frac{k}{2}}\varrho_\varepsilon \mathbf{N}^{\frac{k}{2}}] < C_k < \infty$, for all $k \in \mathbb{N}$. Then the reduced density matrix $\gamma_\varepsilon^{(p)},\ p \in \mathbb{N}$, associated with ϱ_ε is the element of $\mathcal{L}^1(\bigvee^p \mathcal{Z})$ defined by*

$$\mathrm{Tr}\,[\gamma_\varepsilon^{(p)}\tilde{b}] = \mathrm{Tr}\,[\varrho_\varepsilon]\frac{\mathrm{Tr}\,[\varrho_\varepsilon b^{Wick}]}{\mathrm{Tr}\,[\varrho_\varepsilon(|z|^{2p})^{Wick}]}, \tag{2.2.5}$$

with $\gamma_\varepsilon^{(p)} = 0$ if $\mathrm{Tr}\,[\varrho_\varepsilon(|z|)^{Wick}] = 0$.

As we mentioned before, there is a 'dimensional defect of compactness', then we cannot expect the convergence of Wigner measures for symbols in $\mathcal{P}(\mathcal{Z})$. However, we want to understand when this convergence holds for these polynomials. The answer uses the convergence of the reduced density matrices and it is specified in the next Proposition.

Proposition 2.2.19. *Assume that the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}}$ satisfy*

$$\text{(i)}\;\; \mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\},$$

$$\text{(ii)}\;\; \lim_{\mathcal{E}'\ni\varepsilon} \mathrm{Tr}\,[\varrho_\varepsilon b^{Wick}] = \int_{\mathcal{Z}} b(z)\, d\mu(z).$$

Define for $p \in \mathbb{N}$,

$$\gamma_0^{(p)} := \frac{1}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu(z)} \int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu(z).$$

Then for all $p \in \mathbb{N}$, the reduced density matrix γ_ε^p converges to $\gamma_0^{(p)}$ in the $\mathcal{L}^1-$norm.

Hence, there is a strong link between the convergence of reduced density matrices and Wigner measures. Actually the Assumption

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,[\varrho_\varepsilon b^{Wick}] = \int_{\mathcal{Z}} b(z)\, d\mu(z), \forall b \in \mathcal{P}_{alg}(\mathcal{Z}),$$

can be replaced by a condition easier to handle, called condition (**PI**)

$$\forall\alpha \in \mathbb{N},\; \lim_{\varepsilon\to 0} \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\alpha] = \int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^{2\alpha}\, d\mu(z) < +\infty. \tag{\textbf{PI}}$$

This is specified in the following Proposition proved in [11].

Proposition 2.2.20. *For a family of normal states $(\varrho_\varepsilon)_{\varepsilon\in\mathcal{E}'}$ such that $\mathcal{M}(\varrho_\varepsilon,\, \varepsilon \in \mathcal{E}) = \{\mu\}$, the following equivalence holds true*

$$\forall\alpha \in \mathbb{N},\; \lim_{\varepsilon\to 0} \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\alpha] = \int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^{2\alpha}\, d\mu(z) \Longleftrightarrow \lim_{\varepsilon\to 0} \mathrm{Tr}\,[\varrho_\varepsilon b^{Wick}] = \int_{\mathcal{Z}} b(z)\, d\mu(z), \forall b \in \mathcal{P}_{alg}(\mathcal{Z}). \tag{2.2.6}$$

Notice that for a family of normal states $(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|)_{N\in\mathbb{N}^*}$ on $\bigvee^N \mathcal{Z}$, the condition (**PI**) is equivalent to the equality

$$\int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^{2\alpha}\, d\mu(z) = 1.$$

Since the Wigner measure μ is automatically carried on the unit ball of $\mathcal{Z}$ (since the family $(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|)_{N\in\mathbb{N}^*}$ is normal), the previous equality gives a more precise information: the Wigner measure is carried on the unit sphere $\mathcal{S}_{\mathcal{Z}}$, i.e $\mu(\mathcal{S}_{\mathcal{Z}}) = 1$.

2.2.4 Relationship between Wick observables and Wigner measures

Wigner measures are defined through Weyl operators nevertheless it is important for the mean-field problem to draw the link with Wick quantization. Their relationship is clarified by the following Proposition proved in [9, Theorem 6.13] and [9, Corollary 6.14].

Proposition 2.2.21. *Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\bigvee^N \mathcal{Z}$ satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle \leqslant CN\,,$$

and

$$\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, N \in \mathbb{N}) = \{\mu\}.$$

Then, for any $b \in \oplus^{alg}_{p,q\geqslant 0} \mathcal{P}^\infty_{p,q}(\mathcal{Z})$,

$$\lim_{\substack{N\to+\infty\\ \varepsilon N=1}} \langle \Psi^{(N)}, b^{Wick} \Psi^{(N)} \rangle = \int_{\mathcal{Z}} b(z)\ d\mu(z)\,,$$
$$\lim_{\substack{N\to+\infty\\ \varepsilon N=1}} \langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\, b^{Wick} \Psi^{(N)} \rangle = \int_{\mathcal{Z}} e^{2i\pi\,\mathrm{Re}\,\langle z,\xi\rangle}\, b(z)\ d\mu(z).$$

The following a priori estimate is a consequence of [12, Proposition 3.11], [12, Lemma 3.13], [11, Lemma 2.14] and [12, Lemma 3.12].

Proposition 2.2.22. *Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ a sequence of normal states on $\bigvee^N \mathcal{Z}$ satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle \leqslant CN\,,$$

and

$$\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, N \in \mathbb{N}) = \{\mu\}.$$

Then the Wigner measure μ is carried by $Q(A)$ (i.e.: $\mu(Q(A)) = 1$) and its restriction to $Q(A)$ is a Borel probability measure on $(Q(A), \|\cdot\|_{Q(A)})$ fulfilling

$$\int_{\mathcal{Z}} \|z\|^2_{Q(A)}\, d\mu(z) \leqslant C\,,$$
$$\text{and} \qquad \mu(B(\mathcal{Z})) = 1\,,$$

where $B(\mathcal{Z})$ is the unit ball of $\mathcal{Z}$.

Some kind of a Fatou's lemma for Wigner measures holds true.

Proposition 2.2.23. *Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\bigvee^N \mathcal{Z}$ satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle \leqslant CN\,,$$

and

$$\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, N \in \mathbb{N}) = \{\mu\}.$$

Then for any $b \in \mathcal{Q}_{p,p}(A)$ such that $\tilde{b} \geqslant 0$,

$$\liminf_{\substack{N\to+\infty\\ \varepsilon N=1}} \langle \Psi^{(N)}, b^{Wick} \Psi^{(N)} \rangle \geqslant \int_{\mathcal{Z}} b(z)\ d\mu(z)\,.$$

Proof. Since $b \in \mathcal{Q}_{p,p}(A)$, $b(z) = \langle z^{\otimes p}, \tilde{b}\, z^{\otimes p}\rangle$ with $\tilde{b} \in \mathcal{L}(\mathfrak{Q}_p, \mathfrak{Q}_p')$, $\tilde{b} \geqslant 0$, then the quadratic form

$$(\Psi, \Phi) \in Q(H_p^0) \times Q(H_p^0) \to \langle \Psi, \tilde{b}\, \Phi\rangle\,,$$

is closed and non-negative. Hence by [99, Theorem VIII] there exists a unique self-adjoint operator on $\vee^p \mathcal{Z}$, denoted by B, such that $\langle \Psi, \tilde{b}\, \Phi\rangle = \langle \Psi, B\, \Phi\rangle$ for any $\Psi, \Phi \in D(B)$ and $D(B)$ is dense in $Q(H_p^0)$. Moreover, the inequality $0 \leqslant B \leqslant c\, H_p^0$ holds in the sense of quadratic forms on $Q(H_p^0) \subset Q(B)$. So, when $\varepsilon = \frac{1}{N}$,

$$\langle \Psi^{(N)}, b^{Wick} \Psi^{(N)}\rangle = \frac{N!}{N^p (N-p)!} \langle \Psi^{(N)}, B \otimes 1^{(N-p)} \Psi^{(N)}\rangle \geqslant \frac{N!}{N^p(N-p)!} \langle \Psi^{(N)}, \chi_m(B) B \otimes 1^{(N-p)} \Psi^{(N)}\rangle\,,$$

where χ_m is a suitable cutoff function such that $0 \leqslant \chi_m \leqslant 1$ and $\chi_m \to 1$ when $m \to \infty$. For any compact operator C on $\vee^p \mathcal{Z}$ satisfying $0 \leqslant C \leqslant \chi_m(B) B$, one get

$$\langle \Psi^{(N)}, b^{Wick} \Psi^{(N)}\rangle \geqslant \frac{N!}{N^p(N-p)!} \langle \Psi^{(N)}, C \otimes 1^{(N-p)} \Psi^{(N)}\rangle\,.$$

So using Proposition 2.2.21 one obtains

$$\liminf_{\substack{N \to \infty \\ \varepsilon N = 1}} \langle \Psi^{(N)}, b^{Wick} \Psi^{(N)}\rangle \geqslant \int_{\mathcal{Z}} \langle z^{\otimes p}, C\, z^{\otimes p}\rangle\, d\mu,$$

for any non-negative compact operator C such that $C \leqslant \chi_m(B) B$. Remark that there exists a sequence of such operators C_k which converges strongly to $\chi_m(B) B$. Therefore using Proposition 2.2.22 and dominated convergence one obtains

$$\liminf_{\substack{N \to \infty \\ \varepsilon N = 1}} \langle \Psi^{(N)}, b^{Wick} \Psi^{(N)}\rangle \geqslant \int_{\mathcal{Z}} \langle z^{\otimes p}, B\, z^{\otimes p}\rangle\, d\mu = \int_{Q(A)} b(z)\, d\mu\,.$$

∎

Chapter 3

Measure valued solutions to Liouville's equation

Liouville's equation is a fundamental equation of statistical mechanics which describes the time evolution of phase-space distribution functions. Consider for instance a Hamiltonian system $H(p,q) = H(p_1,\cdots,p_n,q_1,\cdots,q_n)$ of finite degrees of freedom where $(q_1,\cdots,q_n,p_1\cdots,p_n)$ are the position-momentum canonical coordinates. Then the time evolution of a probability density function $\varrho(p,q,t)$ describing the system at time t is governed by the Liouville equation,

$$\frac{\partial \varrho}{\partial t} + \{\varrho, H\} = 0\,,$$

with the Poisson bracket defined as follows,

$$\{\varrho, H\} = \sum_{i=1}^{n} \left[\frac{\partial H}{\partial p_i}\frac{\partial \varrho}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial \varrho}{\partial p_i} \right].$$

By formally differentiating $\varrho(p_t,q_t,t)$ with respect to time, where (p_t,q_t) are solutions of the Hamiltonian equations, we recover the Liouville's theorem as stated by Gibbs "The distribution function is constant along any trajectory in phase space", i.e.,

$$\frac{d}{dt}\varrho(p_t,q_t,t) = 0\,.$$

In fact the characteristics method says that if the Hamiltonian is sufficiently smooth and generates a unique Hamiltonian flow Φ_t on the phase-space, then the density function $\varrho(p,q,t)$ is uniquely determined by its initial value $\varrho(p,q,0)$ and it is given as the propagation along the characteristics, i.e.,

$$\varrho(p,q,t) = \varrho(\Phi_t^{-1}(p,q),0)\,.$$

It is known that Liouville's theorem holds in more general framework than the Hamiltonian systems. Consider a differential equation,

$$\frac{d}{dt}X = F(X), \quad X(t=0) = X_0\,, \tag{3.0.1}$$

with $X = (X_1, \cdots, X_n) \in \mathbb{R}^n$ and $F = (F_1, \cdots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth vector field such that a unique flow map $\Phi_t : \mathbb{R}^n \mapsto \mathbb{R}^n$ exists and solves the ODE (3.0.1). If the system (3.0.1) is at an initial statistical state described by a probability density function $\varrho(X, 0)$ at $t = 0$, then under the flow map Φ_t, the evolution of this state is described by a density $\varrho(X, t)$, which is the pull-back of the initial one,

$$\varrho(X, t) = \varrho(\Phi_t^{-1}(X), 0)\,. \tag{3.0.2}$$

If the vector field F satisfies the Liouville's property, which is the following divergence-free condition,

$$\operatorname{div}(F) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial X_j} = 0\,,$$

then the flow map Φ_t is volume preserving (or measure preserving) on the phase space and for all times the density $\varrho(X, t)$ verifies the Liouville equation,

$$\frac{\partial \varrho}{\partial t} + F \cdot \nabla_X \varrho = 0\,. \tag{3.0.3}$$

Again when the vector field is smooth the characteristics theory says that (3.0.2) is the unique solution of the Liouville equation (3.0.3) with the initial value $\varrho(X, 0)$. This enlightens the relationship between individual solutions of the ODE (3.0.1) and statistical (probability measure) solutions of the Liouville equation (3.0.3) and suggests that this is a general principle that could extend to non-smooth vector fields or to dynamical systems with infinite degrees of freedom. Actually the non-smooth framework has been carefully studied and uniqueness of probability measure solutions of Liouville's equation is established via a general superposition principle, see [5, 3, 23, 33, 89, 97] and also [17, 32]. The extension to dynamical systems with infinite degrees of freedom is less studied and the investigations are not oriented toward the study of classical PDEs, see [4, 70, 111], at the exception of the work [12, Appendix C] where the ideas of [5] was adapted to a rigged Hilbert space and applied to the nonlinear Hartree equation with singular potential.

Our aim in this article is to consider the above uniqueness property for Hamiltonian systems with infinite degrees of freedom related to some interesting nonlinear PDEs like the wave or Schrödinger equations. Beyond the fact that Liouville's equation is a natural ground for a statistical theory of Hamiltonian PDEs, we do have another motivation when addressing the previous uniqueness problem. In fact, when we study the relationship between quantum field theories and classical PDEs we encounter the above uniqueness problem, see [7, 12, 13]. Roughly speaking, the quantum counterpart of Liouville's equation is the Von Neumann equation which describes the time evolution of quantum states of Hamiltonian (linear) systems. If we attempt to carry on the classical limit, i.e. $\hbar \to 0$ where $\hbar$ is an effective Planck constant which depends on the scaling of the system at hand, then quantum states transform in the limit into probability measures satisfying a Liouville equation related to a nonlinear Hamiltonian PDE, see [9, 10, 11]. Therefore the uniqueness property for probability measure solutions of Liouville's equation will be a crucial step towards a rigourous justification of the classical limit or the so-called Bohr's correspondence principle.

It is not so obvious how to generalize the above considerations to Hamiltonian systems with infinite degrees of freedom [111]. One of the difficulties for instance is the lack of translation-invariant measures on infinite dimensional normed spaces. Nevertheless, the approach elaborated in [5] is well suited to a generalization for systems with infinite degrees of freedom. This was accomplished in [12] with the following Liouville's equation considered in a weak sense,

$$\partial_t \mu_t + \nabla^T (F.\mu_t) = 0\,, \tag{3.0.4}$$

where $t \mapsto \mu_t$ are probability measure solutions and F is a non-autonomous vector field defined on a rigged Hilbert space $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}_1'$. The result on uniqueness of probability measure solutions of Liouville's (3.0.4) proved in [12, Appendix C] uses a slightly strong assumption on the vector field F,

$$\forall T > 0, \exists\, C > 0, \quad \int_{-T}^{T} \Big[\int_{\mathcal{Z}_1} ||F(t,z)||^2_{\mathcal{Z}_1} d\mu_t(z) \Big]^{\frac{1}{2}} dt \leqslant C\,. \tag{3.0.5}$$

This result was applied in [7, 8, 12, 78] to the mean-field theory and to the classical limit of quantum field theories. In this article we relax the above condition so that we require only the uniform estimate,

$$\forall T > 0, \exists\, C > 0, \quad \int_{-T}^{T} \int_{\mathcal{Z}_1} ||F(t,z)||_{\mathcal{Z}_0} d\mu_t(z) dt \leqslant C\,, \tag{3.0.6}$$

which fits better the energy method communally used to solve PDEs. To illustrate the difference between (3.0.5) and (3.0.6), we consider the following example. Let $\mathcal{Z}_0 = L^2(\mathbb{R})$, $\mathcal{Z}_1 = H^1(\mathbb{R})$ and consider the nonlinear Schrödinger (NLS) equation,

$$\begin{cases} i\partial_t z_t = -\Delta z_t + |z_t|^2 z_t \\ z_{|t=0} = z_0. \end{cases} \tag{3.0.7}$$

In the interaction representation the NLS equation is equivalent to the PDE,

$$\begin{cases} \partial_t \tilde{z}_t = F(t, \tilde{z}_t) := -ie^{-it\Delta} |e^{it\Delta} \tilde{z}_t|^2 (e^{it\Delta} \tilde{z}_t) \\ z_{|t=0} = z_0. \end{cases} \tag{3.0.8}$$

So, $F : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is interpreted as non-autonomous vector field defined on the energy space $H^1(\mathbb{R})$. Sobolev's embedding gives the existence of a constant $C > 0$ such that

$$||F(t,z)||_{L^2(\mathbb{R})} \leqslant C ||z||^2_{H^1(\mathbb{R})} ||z||_{L^2(\mathbb{R})}\,.$$

So, suppose that we have the following a priori information on the measures $(\mu_t)_{t\in\mathbb{R}}$,

$$\int_{H^1(\mathbb{R})} ||z||^2_{H^1(\mathbb{R})}\, ||z||_{L^2(\mathbb{R})}\, d\mu_t \leqslant C \tag{3.0.9}$$

for some time-independent constant C, then the assumption (3.0.6) is satisfied. The requirement (3.0.9) says in some sense that μ_t has a finite energy and actually this can proved a priori, see [12]. However, if we examine (3.0.5) in this case, we see that $||F(t,z)||_{H^1(\mathbb{R})}$ is bounded by

$$||F(t,z)||_{H^1(\mathbb{R})} \leqslant C ||z||^3_{H^1(\mathbb{R})}\,,$$

and hence we need a stronger a priori estimate

$$\int_{H^1(\mathbb{R})} ||z||^6_{H^1(\mathbb{R})}\, d\mu_t \leqslant C\,,$$

which in contrast is difficult to prove a priori. In conclusion, the improvement provided in this article allows to show general and stronger results in the mean-field theory of quantum many-body dynamics, see [8]. The proof of our main Theorem 3.1.1 is based on the work of S. Magnilia [89], Z. Ammari and F. Nier [12] and L. Ambrosio, N. Gigli and G. Savaré [5].

3.1 Result

In this section we define the Liouville equation in a separable infinite dimensional Hilbert space $\mathcal{Z}$. We denote $\mathfrak{P}(\mathcal{Z})$ the set of Borel probability measures on $\mathcal{Z}$. Actually, we consider the equation (3.0.4) in a weak sense that we explain below. The extension of the characteristics theory to systems with infinite degrees of freedom is based on the integration of the equation (3.0.4) after testing by cylindrical test functions. Recall that a function $f : \mathcal{Z} \to \mathbb{C}$ is said cylindrical if there exists a orthogonal projection $\mathfrak{p}$ with finite rank and a function g on $\mathfrak{p}\mathcal{Z}$ such that $f(z) = g(\mathfrak{p}z)$ for all $z \in \mathcal{Z}$. The set of $\mathcal{C}_0^\infty$-cylindrical functions on $\mathcal{Z}$ is denoted $\mathcal{C}_{0,cyl}^\infty(\mathcal{Z})$. So, we can define properly a weak Liouville equation by integrating against test functions on the space $\mathcal{C}_{0,cyl}^\infty(\mathbb{R} \times \mathcal{Z})$. In this context, the velocity fields are singular and the characteristics theory cannot be applied directly. Moreover, the lack of compactness on balls of $\mathcal{Z}$ induces the choice of a topology on $\mathcal{Z}$ which is weaker than the strong one. Introduce $(e_n)_{n\in\mathbb{N}^*}$ a Hilbert basis on $\mathcal{Z}$ and define the topology $(\mathcal{Z}, d_{w,\mathcal{Z}})$ induced by the following distance

$$d_{w,\mathcal{Z}}(z_1 - z_2) = \sqrt{\sum_{n\in\mathbb{N}^*} \frac{|\langle z_1 - z_2, e_n\rangle_{\mathcal{Z}}|^2}{1+n^2}},\ z_1, z_2 \in \mathcal{Z}.$$

We will consider Borel probability measures, solution of the PDE (3.1.1), that are narrowly continuous for this weak topology. In the sequel, we talk about 'weak narrowly continuous' solution on $\mathcal{Z}$ of (3.1.1) (refering to the narrow convergence for continuous bounded test functions on $(\mathcal{Z}, d_{w,\mathcal{Z}})$).
We also denote for $T > 0$, $\Gamma_T(\mathcal{Z})$ the space of continuous maps from $[0,T]$ into $\mathcal{Z}$ equipped with the 'sup' norm. Denote also $AC([0,T], \mathcal{Z})$ the space of absolutely continuous curves on $\mathcal{Z}$ with $L^1([0,T]; \mathcal{Z})$ derivative.

Theorem 3.1.1. *Let $\mathcal{Z}_0$ be a separable Hilbert space. Denote $\mathcal{Z}_1$ a dense subset such that we have a rigged Hilbert space $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}_1'$. Let $\mu_t : \mathbb{R} \mapsto \mathfrak{P}(\mathcal{Z}_0)$ be a weakly narrowly continuous on $\mathcal{Z}_0$, solution of the equation:*

$$\partial_t\mu_t + \nabla^T(v_t\mu_t) = 0,$$

in the weak sense

$$\int_{\mathbb{R}}\int_{\mathcal{Z}_0} \partial_t\phi(t,z) + \operatorname{Re}\langle v_t(z), \nabla_z\phi(t,z)\rangle_{\mathcal{Z}_0}\, d\mu_t(z)dt = 0, \forall\phi \in \mathcal{C}_{0,cyl}^\infty(\mathbb{R}\times\mathcal{Z}_0), \tag{3.1.1}$$

for a suitable Borel velocity field $v(t,z) = v_t(z)$ such that

$$\int_0^T\int_{\mathcal{Z}_0} \|v_t(z)\|_{\mathcal{Z}_0}\, d\mu_t(z) < \infty,\ \forall T > 0, \tag{3.1.2}$$

and such that the time-dependant measure μ_t is carried on $\mathcal{Z}_1$, and is a Borel probability measure on $\mathcal{Z}_1$. Assume additionally that the Cauchy problem

$$\partial_t\gamma(t) = v_t(\gamma(t)), \gamma(0) = x, \tag{3.1.3}$$

admits a global continuous solution $t \mapsto \gamma_t := \Phi(t,0)x \in \mathcal{C}^0(\mathbb{R}, \mathcal{Z}_1) \cap \mathcal{C}^1(\mathbb{R}, \mathcal{Z}_1')$ where $\Phi(t,0) : \mathcal{Z}_1 \to \mathcal{Z}_1$ is a Borel flow. Then the measure μ_t is the push-forward of the initial measure μ_0 by the flow $\Phi(t,0)$, i.e for all $t \in \mathbb{R}$

$$\mu_t = \Phi(t,0)_*\mu_0. \tag{3.1.4}$$

For the sake of simplicity, we introduce the vocabulary of triple solutions of the equation (3.0.4) in a general infinite dimensional separable Hilbert space $\mathcal{Z}$.

Definition 3.1.2. *We say that a triple* $(\mu_t, v_t, \mathcal{Z})$ *is a solution of* **the weak Liouville equation** *if the equation* (3.1.1) *holds true, where* $v_t(z) : \mathcal{Z} \to \mathcal{Z}$ *is a velocity vector field associated to* (3.1.3) *and satisfying the estimate* (3.1.2).

Remarks 3.1.3. *Let* $(\mu_t, v_t, \mathcal{Z}_0)$ *be a triple solution of the* **weak liouville equation**. *Assume that the measure* μ_t *is carried on the dense subset* $\mathcal{Z}_1 \subset \mathcal{Z}_0$. *Without a loss of generality we can always assume that* $v_t : \mathcal{Z}_0 \to \mathcal{Z}_0$. *In fact for a Borel velocity field* $v_t : \mathcal{Z}_1 \to \mathcal{Z}$ *we set* $\hat{v}_t := v_t$ *on* $\mathcal{Z}_1$ *and* $\hat{v}_t := 0$ *on* $\mathcal{Z}_0 \backslash \mathcal{Z}_1$. *Therefore since* $\forall t \in \mathbb{R}$, $\mu_t(\mathcal{Z}_1) = 1$, *the triple* $(\mu_t, \hat{v}_t, \mathcal{Z}_0)$ *is also solution of the* **weak Liouville equation.**

Definition 3.1.4. *We introduce a new condition, called* **(CP)** *if a triple* $(\eta, v_t, \mathcal{Z} \times \Gamma_T(\mathcal{Z}))$ *satisfies* **(CP)***: the measure* $\eta \in \mathfrak{P}(\mathcal{Z} \times \Gamma_T(\mathcal{Z}))$ *is concentrated on the set of* (x, γ) *with* $\gamma \in AC([0,T]; \mathcal{Z})$ *that are solutions of*

$$\partial_t \gamma(t) = v_t(\gamma(t)), \gamma(0) = x, \tag{3.1.5}$$

with a Borel velocity vector field $v_t : \mathcal{Z} \to \mathcal{Z}$.

Additionally denote the time dependent Borel probability measure $\mu_t^\eta \in \mathfrak{P}(\mathcal{Z})$ defined by

$$\int_{\mathcal{Z}} \varphi \, d\mu_t^\eta = \int_{\mathcal{Z} \times \Gamma_T(\mathcal{Z})} \varphi(\gamma(t)) \, d\eta(x, \gamma), \ \forall \varphi \in \mathcal{C}^0_{b,cyl}(\mathcal{Z}), \ t \in [0,T], \tag{3.1.6}$$

where $\mathcal{C}^0_{b,cyl}(\mathcal{Z})$ is the space of cylindrical bounded continuous functions on $\mathcal{Z}$. The measure μ_t^η is the push-forward of η by the evaluation map

$$e_t : (x, \gamma) \in \mathcal{Z} \times \Gamma_T(\mathcal{Z}) \mapsto \gamma(t) \in \mathcal{Z}, \text{ for } t \in [0,T].$$

Recall we set a rigged Hilbert space $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}_1'$ where $\mathcal{Z}_1$ is a dense subset of $\mathcal{Z}_0$.

Proof. The plan is to project the equation (3.1.1) on a finite dimensional space to get the existence of a measure, which we denote in the sequel by μ_t^d belonging to $\mathfrak{P}(\mathbb{R}^d)$, and a velocity field $v_t^d : \mathbb{R}^d \to \mathbb{R}^d$ such that for $t \in [0,T]$ the triple $(\mu_t^d, v_t^d, \mathbb{R}^d)$ is a solution of the **weak Liouville equation**.

- **Projection to the finite dimension:**
  Consider the Hilbert basis $(e_n)_{n \in \mathbb{N}*}$ of $\mathcal{Z}_0$ and the following diagram

$$\begin{tikzcd} \mathcal{Z}_0 \arrow[r, "\pi^d"] \arrow[dr, "\hat{\pi}^d"'] & \mathbb{R}^d \arrow[d, "\pi^{d,T}"] \\ & \mathcal{Z}_0 \end{tikzcd}$$

  with $\pi^d(x) = (\langle e_1, x\rangle, ..., \langle e_d, x\rangle)$, $\pi^{d,T}(y_1, y_2, ..., y_d) = \sum_{j=1}^d y_j e_j$ and $\hat{\pi}^d = \pi^{d,T} \circ \pi^d$. Hence we define the measure μ_t^d as the push-forward of the measure μ_t by the projection π^d, i.e

$$\mu_t^d := \pi_*^d \mu_t. \tag{3.1.7}$$

Therefore by the disintegration theorem (see Appendix 3.3.1 or for a more general presentation Chapter V in [5]) there exists a family of measures $\{\mu_{t,y}, y \in \mathbb{R}^d\}$ such that we can define two velocity fields v_t^d and $\hat{v}_t{}^d$:

$$v_t^d(y) = \int_{(\pi^d)^{-1}(y)} \pi^d v_t(x) d\mu_{t,y}(x),\ y \in \mathbb{R}^d \tag{3.1.8}$$

$$\hat{v}_t^d(y) = \int_{(\hat{\pi}^d)^{-1}(\hat{\pi}^d y)} \hat{\pi}^d v_t(x) d\mu_{t,\pi^d y}(x),\ y \in \mathcal{Z}_0. \tag{3.1.9}$$

Similarly we also define the measure

$$\hat{\mu}_t^d := \hat{\pi}_*^d \mu_t. \tag{3.1.10}$$

Since $\pi^d, \hat{\pi}^d$ are projections with finite rank, the two maps

$$t \in [0,T] \mapsto \mu_t^d \in \mathfrak{P}(\mathbb{R}^d)\ ;\ t \in [0,T] \mapsto \hat{\mu}_t^d \in \mathfrak{P}(\mathcal{Z}_0)$$

are weakly narrowly continuous. By using Lemma 3.3.2 the triples $(\mu_t^d, v_t^d, \mathbb{R}^d)$ and $(\hat{\mu}_t^d, \hat{v}_t^d, \mathcal{Z}_0)$ are solutions of the **weak Liouville equations** .

- **Result in finite dimension:**
By using [89, Theorem 4.1] to the triple $(\mu_t^d, v_t^d, \mathbb{R}^d)_{t\in[0,T]}$ there exists a triple $(\eta^d, v_t^d, \mathbb{R}^d \times \Gamma_T(\mathbb{R}^d))$ satisfying the condition **(CP)** where the measure η^d belongs to $\mathfrak{P}(\mathbb{R}^d \times \Gamma_T(\mathbb{R}^d))$. Moreover we can define a measure $\hat{\eta}^d \in \mathfrak{P}(\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$ by the following equality $\hat{\eta}^d := (\pi^{d,T} \times \pi^d)_* \eta^d$, i.e

$$\int_{\mathcal{Z}_0\times\Gamma_T(\mathcal{Z}_0)} \varphi(x,\gamma)\, d\hat{\eta}^d(x,\gamma) = \int_{\mathbb{R}^d\times\Gamma_T(\mathbb{R}^d)} f(\pi^{d,T}x, \pi^d\gamma)\, d\eta^d(x,\gamma), \tag{3.1.11}$$

for every function $\varphi \in \mathcal{C}_{b,cyl}^0(\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$. As a consequence of [89, Theorem 4.1], we have the equality for any $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d),\ t \in [0,T]$,

$$\int_{\mathcal{Z}_0} \varphi(\pi^d)\, d\hat{\mu}_t^d = \int_{\mathbb{R}^d} \varphi\, d\mu_t^d = \int_{\mathbb{R}^d\times\Gamma_T(\mathbb{R}^d)} \varphi(\gamma(t))\, d\eta^d = \int_{\mathcal{Z}_0\times\Gamma_T(\mathcal{Z}_0)} \varphi\circ\pi^d(\gamma(t))\, d\hat{\eta}^d. \tag{3.1.12}$$

- **Weak tightness:**

Let us show the weak tightness of the family $(\hat{\eta}^d)_{d\in\mathbb{N}}$ will use two criterions recalled in Appendix 3.3.2. Choose the maps r^1 and r^2 defined on $\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)$ as

$$r^1 : (x,\gamma) \mapsto x \in \mathcal{Z}_0$$

and

$$r^2 : (x,\gamma) \mapsto \gamma - x \in \Gamma_T(\mathcal{Z}_0),$$

hence notice that the map $r = r^1 \times r^2 : \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)$ is proper. The family $(r_*^1\hat{\eta}^d)_{d\in\mathbb{N}}$ is given by the first marginal $(\hat{\mu}_0^d)_{d\in\mathbb{N}}$. Besides the measure $\hat{\mu}_0^d$ is weakly tight since $\hat{\mu}_0^d \to \hat{\mu}_0$ weakly narrowly. For the family $(r_*^2\hat{\eta}^d)_{d\in\mathbb{N}}$, since the functional

$$g \mapsto \int_0^T |\dot{g}(t)|dt,$$

defined on $\{g \in \Gamma_T(\mathcal{Z}_0), g(0) = 0\}$ and set to $+\infty$ if $g \neq AC([0,T], \mathcal{Z}_0)$, has compact sublevel sets in $\Gamma_T(\mathcal{Z}_0)$ but is not coercive in $\Gamma_T(\mathcal{Z}_0)$, the proof of the weak tightness differs from [12, Proposition C.2].
Assume that there exists a convex superlinear function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the functional

$$\gamma \mapsto \int_0^T \Psi(|\dot{\gamma}(t)|)dt$$

is coercive in $\{g \in \Gamma_T(\mathcal{Z}_0), g(0) = 0\}$. Then by using Lemma [89, 3.10] we get the existence of a regularized triple $(\mu^d_{t,\varepsilon}, v^d_{t,\varepsilon}, \mathbb{R}^d)_{\varepsilon \geqslant 0}$ that is a solution of the **weak Liouville equation** such that for $t \in [0,T]$

1. there exists a unique maximal solution $X_\varepsilon(t,s,\pi^d x)$ of $\frac{d}{dt}X_\varepsilon(t,s,.) = v^d_{t,\varepsilon}(X_\varepsilon(t,s,.))$ with $X_\varepsilon(s,s,\pi^d x) = \pi^d x$,
2. $\mu^d_{t,\varepsilon} = X(t,0,.)_* \mu^d_{0,\varepsilon}$,
3. there exist two families of measures $(\eta^d_\varepsilon)_{\varepsilon \geqslant 0}$, $(\hat{\eta}^d_\varepsilon)_{\varepsilon \geqslant 0}$ given by the following equalities

$$\eta^d_\varepsilon = (\text{ Id } (x) \times X_\varepsilon(t,0,x))_* \mu^d_{0,\varepsilon}$$

$$\hat{\eta}^d_\varepsilon := (\pi^{d,T} \times \pi^d)_* \eta^d_\varepsilon; \tag{3.1.13}$$

4. for any $T > 0$

$$\int_0^T \int_{\mathbb{R}^d} \Psi(|v^d_{t,\varepsilon}(x)|)\, d\mu^d_{t,\varepsilon}(x)\, dt \leqslant \int_0^T \int_{\mathbb{R}^d} \Psi(|v^d_t(x)|)\, d\mu^d_t(x)\, dt. \tag{3.1.14}$$

5. For $d \in \mathbb{N}$ the family $(\hat{\eta}^d_\varepsilon)_{\varepsilon \geqslant 0}$ is weakly tight.

Therefore we compute the quantity

$$\begin{aligned}
\int_{\Gamma_T(\mathcal{Z}_0)} \int_0^T \Psi(|\dot{\gamma}(t)|)dt\, d((r_2)_* \hat{\eta}^d_\varepsilon)(\gamma) &= \int_{\mathbb{R}^d} \int_0^T \Psi(|\dot{X}_\varepsilon(t,0,x)|)\, dt\, d\mu^d_{0,\varepsilon}(x) \\
&\leqslant \int_0^T \int_{\mathbb{R}^d} \Psi(|v^d_{t,\varepsilon}(x)|)\, d\mu^d_{t,\varepsilon}(x)\, dt \\
&\leqslant \int_0^T \int_{\mathbb{R}^d} \Psi(|v^d_t(x)|)\, d\mu^d_t(x)\, dt \\
&\leqslant \int_0^T \int_{\mathcal{Z}_0} \Psi(\|v_t(x)\|_{\mathcal{Z}_0})\, d\mu_t(x)\, dt < \infty,
\end{aligned}$$

the last step coming from the estimate (3.3.5) in Lemma 3.3.2. Since the family $(\hat{\eta}^d_\varepsilon)_{\varepsilon \geqslant 0}$ is weakly tight, take the limit in the l.h.s when $\varepsilon \to 0$ to get

$$\int_{\Gamma_T(\mathcal{Z}_0)} \int_0^T \Psi(|\dot{\gamma}(t)|)dt\, d((r_2)_* \hat{\eta}^d)(\gamma) < +\infty. \tag{3.1.15}$$

Now, it remains to prove the existence of the function Ψ. By using the Dunford-Pettis theorem, the uniform control (3.1.2) leads to the existence of superlinear convex function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_0^T \int_{\mathcal{Z}_0} \Psi(\|v_t(x)\|_{\mathcal{Z}_0})\, d\mu_t(x) dt < +\infty.$$

Indeed set

$$\mu := \int_0^T \mu_t\, dt, \text{ hence } \int_0^T \int_{\mathcal{Z}_0} \|v_t\|_{\mathcal{Z}_0}\, d\mu_t(x)dt = \int_{\mathcal{Z}_0} \|v\|_{\mathcal{Z}_0}\, d\mu,$$

since the family $\{v\}$ is a compact set. Then the family $((r_2)_* \hat{\eta}^d)_{d\in\mathbb{N}}$ is weakly tight.
Let us sketch the rest of the proof by using diagrams for triples introduced in the previous steps. By Lemma 3.3.2 we deduce the following diagram

$$\begin{tikzcd}
(\mu_t, v_t, \mathcal{Z}_0) \arrow[r] \arrow[dr] & (\mu_t^d, v_t^d, \mathbb{R}^d) \arrow[d] \\
& (\hat{\mu}_t^d, \hat{v}_t^{\,d}, \mathcal{Z}_0)
\end{tikzcd}$$

where each triple is a solution of the **weak Liouville equation**. By using the **weak tightness** step to the sequence $(\hat{\eta}^d)_{d\in\mathbb{N}}$, it gives rise to a probability measure $\eta \in \mathfrak{P}(\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$. Similarly as in the two previous diagrams we shall complete the following diagram for triples satisfying the **(CP)** condition:

$$\begin{tikzcd}
(\eta, v_t, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)) & (\eta^d, v_t^d, \mathbb{R}^d \times \Gamma_T(\mathbb{R}^d)) \arrow[l] \arrow[d] \\
& (\hat{\eta}^d, \hat{v}_t^{\,d}, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)) \arrow[ul]
\end{tikzcd}$$

Therefore in the following step we shall prove the existence of the triple $(\eta, v_t, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$ satisfying the **(CP)** condition.

- **Existence the measure concentrated on the solutions of the ODE:**
  We have constructed two triples of probability measures $(\eta^d, v_t^d, \mathbb{R}^d \times \Gamma_T(\mathbb{R}^d))$ and $(\hat{\eta}^d, \hat{v}_t^{\,d}, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$ such that the sequence $(\hat{\eta}^d)_{d\in\mathbb{N}}$ defined in (3.1.11) is weakly tight. Therefore set η as a narrow limit point of $\hat{\eta}^d$. Assume that the test function φ in (3.1.12) depends only on d' coordinates with $d' \leqslant d$. Hence taking the limit when $d \to +\infty$ gives

$$\int_{\mathcal{Z}_0} \varphi \circ \pi^{d'} d\mu_t = \int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} (\varphi \circ \pi^{d'})(\gamma(t))\, d\eta(x, \gamma),$$

  for all $\varphi \in \mathcal{C}_b^0(\mathbb{R}^{d'})$ and $t \in [0, T]$, where $\varphi \circ \pi^{d'}$ can be replaced by any cylindrical function or Borel bounded function on $\mathcal{Z}_0$.

- **The concentration condition, (CP):**
  We shall prove the following equality for $t \in [0, T]$

$$\int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \|\gamma(t) - x - \int_0^t v_s(\gamma(s))\, ds\|_{\mathcal{Z}_0}\, d\eta(x, \gamma) = 0. \tag{3.1.16}$$

In the finite dimensional case, by a regularization process like in Step 5 of Theorem [89, 4.1], there exists a sequence $(v^d_{t,n})_{n\in\mathbb{N}}$ of uniformly continuous fonction in $\mathcal{C}^0_b([0,T]\times\mathcal{Z}_0;\mathcal{Z}_0)$ such that $\|v^d_t - v^d_{t,n}\|_{L^1(\mathbb{R}^d,d\mu^d_t)} \to 0$ and

$$\int_{\mathbb{R}^d\times\Gamma_T(\mathbb{R}^d)} |\gamma(t) - x - \int_0^t v^d_{s,n}(\gamma(s))\,ds|\,d\eta^d(x,\gamma) \leqslant \int_0^T \int_{\mathbb{R}^d} |v^d_s - v^d_{s,n}|\,d\mu^d_s\,ds. \tag{3.1.17}$$

The equality (3.1.16) can be deduced from the finite dimensional case. Indeed set the function w_t belonging to $\mathcal{C}^0_b([0,T]\times\mathbb{R}^{d'};\mathbb{R}^{d'})$ with $d'\leqslant d$ fixed and, by setting $\hat{w}_t = \pi^{d',T}\circ w_t\circ\pi^{d'}\in \mathcal{C}^0_b([0,T]\times\mathcal{Z}_0;\mathcal{Z}_0)$, we get

$$\begin{aligned}
&\int_{\mathcal{Z}_0\times\Gamma_T(\mathcal{Z}_0)} \|\gamma(t) - x - \int_0^t v_s(\gamma(s))\,ds\|_{\mathcal{Z}_0}\,d\eta(x,\gamma) \leqslant \\
&\underbrace{\int_{\mathcal{Z}_0\times\Gamma_T(\mathcal{Z}_0)} \|\gamma(t) - x - \int_0^t \hat{w}_s(\gamma(s))\,ds\|_{\mathcal{Z}_0}\,d\eta(x,\gamma)}_{\mathcal{A}(\hat{w})} \\
&+\underbrace{\int_{\mathcal{Z}_0\times\Gamma_T(\mathcal{Z}_0)} \int_0^t \|\hat{w}_s(\gamma(s)) - v_s(\gamma(s))\|_{\mathcal{Z}_0}\,ds\,d\eta(x,\gamma)}_{\mathcal{B}(\hat{w})}.
\end{aligned}$$

The first term on the r.h.s, $\mathcal{A}(\hat{w})$ can be estimated owing to finite dimensional estimate (3.1.17). Indeed it follows by the same regularization process presented in (3), the next estimate holds true

$$\mathcal{A}(\hat{w}) \leqslant \limsup_{d\to+\infty} \int_0^T \int_{\mathcal{Z}_0} \|\hat{v}^d_s - \hat{w}_s\|_{\mathcal{Z}_0}\,d\hat{\mu}^d_s\,ds \leqslant \int_0^T \|v_s - \hat{w}_s\|_{L^1(\mathcal{Z}_0,d\mu_s)}\,ds,$$

by using the estimate (3.3.4) in Lemma 3.3.2 to the function $\hat{v}^d_s - \hat{w}_s$. We conclude the proof by noticing that there exists a sequence of cylindrical, uniformly bounded continuous bounded function $(\hat{w}^n_s)_{n\in\mathbb{N}}$ such that for any $t\in[0,T]$ $\|v_t - \hat{w}^n_t\|_{L^1(\mathcal{Z}_0,d\mu_t)} \longrightarrow 0$. Therefore $\mathcal{A}(\hat{w}^n), \mathcal{B}(\hat{w}^n) \longrightarrow 0$ and the equality (3.1.16) is proved.

- **End of the proof:** The relation $\mu_t = \mu^\eta_t$ defined according to (3.1.12) extends to any bounded Borel function φ on $\mathcal{Z}_0$

$$\int_{\mathcal{Z}_0} \varphi\,d\mu_t = \int_{\mathcal{Z}_1} \varphi\,d\mu_t = \int_{\mathcal{Z}_1\times\Gamma_T(\mathcal{Z}_1)} \varphi(\gamma(t))\,d\eta(x,\gamma),$$

since the measure μ_t is carried on $\mathcal{Z}_1$. In particular this relation is true when $t=0$, with a function $\varphi\in C^\infty_c(\mathcal{Z}_0)$ such that $\operatorname{supp}(\varphi)\subset\mathcal{Z}_0\backslash\mathcal{Z}_1$

$$\int_{\mathcal{Z}_1} \varphi\,d\mu_0 = 0 = \int_{\mathcal{Z}_1\times\Gamma_T(\mathcal{Z}_1)} \varphi(\gamma(0))\,d\eta(x,\gamma).$$

Hence

$$\eta(\{(x,\gamma)\ /\ t\in\mathbb{R},\ \partial_t\gamma(t) = v_t(\gamma(t)), \gamma(0) = x,\ x\notin\mathcal{Z}_1\}) = 0,$$

and by using the Cauchy problem uniqueness

$$\eta(\{(x,\gamma) \ / \, t \in \mathbb{R}, x \in \mathcal{Z}_1, \, \gamma(t)(x) = \Phi(t,0)x\}) = 1,$$

where $\Phi(t,0) : \mathcal{Z}_1 \to \mathcal{Z}_1$ is a Borel flow associated to the ODE (3.1.3), we deduce for any $t \in \mathbb{R}$

$$\int_{\mathcal{Z}_0} \varphi \, d\mu_t = \int_{\mathcal{Z}_1 \times \Gamma_T(\mathcal{Z}_1)} [\varphi \circ \Phi(t,0)](\gamma(0)) \, d\eta(x,\gamma) = \int_{\mathcal{Z}_1} [\varphi \circ \Phi(t,0)] \, d\mu_0,$$

which ends the proof.

∎

3.2 Examples

We illustrate our main Theorem 3.1.1 with few examples. Consider a semi-linear Hamiltonian PDEs with a (real-valued) energy functional having the form,

$$h(z,\bar{z}) = \langle z, Az \rangle_{\mathcal{Z}_0} + h_I(z,\bar{z}) \,,$$

where $\mathcal{Z}_0$ is a complex Hilbert space, A is a non-negative self-adjoint operator, $h_I(z,\bar{z})$ is a nonlinear functional and $(z,\bar{z})$ are the complex classical fields of the Hamiltonian theory. So that the related PDE (or equation of motion) is,

$$i\partial_t u = Au + \partial_{\bar{z}} h_I(u,\bar{u}) \,. \tag{3.2.1}$$

By differentiating $\tilde{u} := e^{itA}u$ with respect to time, we equivalently express the above equation in the interaction representation, i.e.,

$$\partial_t \tilde{u} = -ie^{itA} \partial_{\bar{z}} h_I(e^{-itA}\tilde{u}, \overline{e^{-itA}\tilde{u}}) \,.$$

Hence the original PDE (3.2.1) can be reformulated as an ODE,

$$\frac{d}{dt} u = v(t,u) \,,$$

with a non-autonomous vector field $v(t,\cdot)$ given by

$$v(t,z) := -ie^{itA} \partial_{\bar{z}} h_I(e^{-itA}z, \overline{e^{-itA}z}) \,.$$

The natural energy space is $Q(A) := D(A^{\frac{1}{2}})$, the form domain of A equipped with the graph norm,

$$||z||_{Q(A)}^2 = \langle z, (A+1)\, z \rangle_{\mathcal{Z}_0} \,.$$

In all the examples considered below we have that $v(\cdot,\cdot) : \mathbb{R} \times Q(A) \to \mathcal{Z}_0$ is a continuous map satisfying the following estimate (or a similar one),

$$||v(t,z)||_{\mathcal{Z}_0} \leqslant C ||z||_{Q(A)}^2 \, ||z||_{\mathcal{Z}_0} \,,$$

for some time-independent constant $C > 0$. Moreover, we have that the energy $h(z,\bar{z})$ makes sense on the space $Q(A)$ and the Cauchy-problem (3.2.1) is globally well-posed on $Q(A)$ in the sense of existence and uniqueness of a global strong solution $t \mapsto z(t) \in \mathcal{C}^0(\mathbb{R}, Q(A)) \cap \mathcal{C}^1(\mathbb{R}, Q'(A))$ for each $z_0 \in Q(A)$ and continuous dependence on initial data.

Example 3.2.1 (The nonlinear Schrödinger equation)**.** *The energy functional of the NLS equation in dimension* $d = 1$ *is,*

$$h(z, \bar{z}) = \langle z, -\Delta_x + V(x)\, z\rangle_{L^2(\mathbb{R}^d)} + \frac{\lambda}{2}\int_{\mathbb{R}} |z(x)|^4 \, dx\,, \tag{3.2.2}$$

where V *is a real-valued potential which splits into a positive and negative part* $V = V_+ + V_-$ *such that* $V_+ \in L^1_{loc}(\mathbb{R})$ *and* V_- *is* $-\Delta$*-form bounded with a relative bound less than one. So the quadratic form* $A = -\Delta + V$ *defines a self-adjoint operator semi-bounded from below and its natural domain* $Q(A)$ *is a Hilbert space when equipped with the graph norm,*

$$||u||^2_{Q(A)} = \langle u, (A + V_+ + 1)\, u\rangle\,.$$

The vector field in this case is $v(t, z) = 2\lambda|e^{-itA}z|^2\, e^{-itA}z : \; Q(A) \to L^2(\mathbb{R})$ *and satisfies the following inequalities for all* $z \in Q(A)$*,*

$$\|v(t, z)\|_{L^2(\mathbb{R})} \leqslant C\|z\|^2_{H^1(\mathbb{R})}\|z\|_{L^2(\mathbb{R})} \leqslant C\|z\|^2_{Q(A)}\|z\|_{L^2(\mathbb{R})}\,, \tag{3.2.3}$$

since the inclusion $Q(A) \subset H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ *holds by Sobolev's embedding and the fact that* $Q(A) = \{u \in L^2(\mathbb{R}), u' \in L^2(\mathbb{R}), V_+^{\frac{1}{2}}u \in L^2(\mathbb{R})\}$*. Moreover, it is known that the NLS equation*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + \lambda|z|^2 z \\ z_{|t=0} = z_0, \end{cases} \tag{\textbf{NLS}}$$

is globally well-posed on $Q(A)$ *with energy and charge conservation. The derivation of such equation from quantum many-body dynamics is established for instance in [1, 6].*

Example 3.2.2 (Non-relativistic Hartree equation)**.** *The energy functional of the Hartree equation is*

$$h(z, \bar{z}) = \langle z, -\Delta_x + V(x)\, z\rangle_{L^2(\mathbb{R}^d)} + \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} |z(x)z(y)|^2\, W(x - y)\, dxdy\,, \tag{3.2.4}$$

where $W : \mathbb{R}^d \to \mathbb{R}$ *is an even measurable function and* V *is a real-valued potential both satisfying the following assumptions for some* p *and* q*,*

$$\begin{aligned} &V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\; p \geqslant 1,\;\; p > \frac{d}{2}, \\ &W \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\; q \geqslant 1,\;\; q \geqslant \frac{d}{2}\;(\textit{and } q > 1 \textit{ if } d = 2)\,. \end{aligned}$$

The vector field $v(t, z) := W * |z|^2 z : \; Q(A) \to L^2(\mathbb{R}^d)$ *verifies the estimate,*

$$\|W * |z|^2 z\|_{L^2(\mathbb{R}^d)} \leqslant \;\|(-\Delta + 1)^{-\frac{1}{2}}W(-\Delta + 1)^{-\frac{1}{2}}\|\;\|z\|^2_{H^1(\mathbb{R}^d)}\;\|z\|_{L^2(\mathbb{R}^d)}\,. \tag{3.2.5}$$

The global well-posedness on $Q(A)$*, conservation of energy and charge of the Hartree equation*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{t=0} = z_0, \end{cases}$$

are proved in [28] Corollary 4.3.3 and Corollary 6.1.2.
We remark that the assumption on W are satisfied by the Coulomb type potentials $\frac{\lambda}{|x|^\alpha}$ when $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$. The derivation of such equation from quantum many-body dynamics is extensively investigated, see for instance [12, 19, 44, 47, 52, 53, 63, 69, 110].

Example 3.2.3 (Semi-relativistic Hartree equation)**.** *The semi-relativistic Hartree equation has the energy functional*

$$h(z,\bar{z}) = \langle z, \sqrt{-\Delta_x + m^2} + V(x)\, z\rangle_{L^2(\mathbb{R}^3)} + \lambda \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|z(x)z(y)|^2}{|x-y|}\, dxdy\,,$$

with $0 \leqslant \lambda$, $m \geqslant 0$ and V is real-valued measurable function which splits into a positive and negative part $V = V_+ + V_-$ satisfying,

$$\begin{aligned} &V_+ \in L^1_{loc}(\mathbb{R}^3)\,,\\ &V_- \textit{ is } \sqrt{-\Delta} - \textit{form bounded with a relative bound less than } 1\,. \end{aligned}$$

The quadratic form

$$\begin{aligned} &A[u,u] = \langle u, \sqrt{-\Delta + m^2}\, u\rangle + \langle u, V u\rangle\,,\\ &Q(A) = \{u \in L^2(\mathbb{R}^3),\ (-\Delta + m^2)^{\frac{1}{4}} u \in L^2(\mathbb{R}^3),\ V_+^{\frac{1}{2}} u \in L^2(\mathbb{R}^3)\}\,, \end{aligned}$$

is semi-bounded from below and closed. So it defines a unique self-adjoint operator denoted by A. Thanks to a Hardy type inequality (see for instance [12, Proposition D.3]), we have the bound,

$$\|\frac{1}{|x|} * |z|^2\, z\|_{L^2(\mathbb{R}^3)} \leqslant C \|z\|^2_{H^{1/2}(\mathbb{R}^3)}\, \|z\|_{L^2(\mathbb{R}^3)}\,.$$

The global well-posedness in $Q(A)$, conservation of energy and charge of the semi-relativistic Hartree equation

$$\begin{cases} i\partial_t z = \sqrt{-\Delta + m^2}\; z + V(x) z + \dfrac{\lambda}{|x|} * |z|^2 z \\ z_{|t=0} = z_0. \end{cases}$$

are proved in [73, Theorem 4] for all $\lambda \geqslant 0$.

Example 3.2.4 (The Klein-Gordon equation)**.** *The classical energy functional formally associated with the quantum field theory $P(\varphi)_2$ is given by*

$$h(z,\bar{z}) := \langle z, Az\rangle_{L^2(\mathbb{R})} + G(z)\,,$$

where A is a multiplication operator by the function $\omega(k) = \sqrt{m_0^2 + k^2}$, $m_0 > 0$, and G is a polynomial interaction defined as follows, see [104, 106]. Consider a bounded from below real polynomial

$$P(x) = \sum_{j=0}^{2n} \alpha_j x^j, \quad (\alpha_{2n} > 0).$$

Let $\varphi(x)$ be the scalar-field of mass $m_0 > 0$, i.e.:

$$\varphi(x) := \int_{\mathbb{R}} e^{-ikx} \left[\bar{z}(k) + z(-k)\right] \frac{dk}{\sqrt{\omega(k)}} ,$$

where $(z, \bar{z})$ are scalar complex fields. Let g a non-negative function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g(x) = g(-x)$. The nonlinear term G is defined as the following real-valued polynomial

$$G(z) := \int_{\mathbb{R}} g(x)\, P\Big(2 \ \mathrm{Re}\, \langle z, \frac{e^{-ikx}}{\sqrt{\omega(k)}}\rangle_{L^2(\mathbb{R})}\Big)\, dx.$$

So that we have at hand the nonlinear Klein-Gordon equation with a non-local nonlinearity,

$$i\partial_t \varphi = \omega\varphi + \partial_{\bar{z}} G(\varphi) . \tag{3.2.6}$$

The local Cauchy problem is studied for instance in [98, Theorem 1] and [100, Theorem X.72]. Actually, one can prove the energy conservation and hence global well-posedness holds true in this specific case. Moreover, the vector field $v(t, z) = -ie^{itA}\partial_{\bar{z}} G(e^{-itA} z) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is continuous and satisfies,

$$||v(t, z)||_{L^2(\mathbb{R})} \leqslant C(1 + ||z||^{2n-1}_{L^2(\mathbb{R})}) .$$

The derivation of such PDE from $P(\varphi)_2$ quantum field theory is established in [13, 36, 63].

Example 3.2.5 (The Schrödinger-Klein-Gordon system)**.** *The Schrödinger-Klein-Gordon system with Yukawa interaction is defined by:*

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M} u + Au \\ (\square + m^2)A = -|u|^2 \end{cases} ; \tag{S-KG}$$

where (u, A) are the unknowns and $M, m > 0$ are real parameters. If we introduce the complex field α, defined by

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{1}{\sqrt{2\omega(k)}} \big(\bar{\alpha}(k) e^{-ik\cdot x} + \alpha(k) e^{ik\cdot x}\big) dk \ , \quad \omega(k) = \sqrt{k^2 + m^2} \, , \tag{3.2.7}$$

we can rewrite (S-KG) *as the equivalent system:*

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M} u + Au \\ i\partial_t \alpha = \omega\alpha + \dfrac{1}{\sqrt{2\omega}} \mathcal{F}\big(|u|^2\big) \end{cases} , \tag{S-KG$_\alpha$}$$

where $\mathcal{F}$ denotes the Fourier transform. It is known that the Cauchy problem for the Schrödinger-Klein-Gordon system (S-KG) *is globally well posed on the energy space, see for instance [31, 94] and references therein. In particular, there is a unique Hamiltonian flow Φ_t for* (S-KG$_\alpha$) *on the energy space $H^1(\mathbb{R}^3) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$ where $\mathcal{F}H^s(\mathbb{R}^d)$ denotes the Fourier Sobolev space,*

$$\mathcal{F}H^s(\mathbb{R}^d) = \Big\{ f \, , \, \mathcal{F}^{-1} f \in H^s(\mathbb{R}^d) \Big\} \, .$$

Moreover, the vector field $v(t,.) : H^1(\mathbb{R}^3) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ satisfies by Gagliardo-Nirenberg's inequality,

$$||v(t, u \oplus \alpha)||_{L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)} \leqslant C(||u||^2_{H^1(\mathbb{R}^3)} + ||\alpha||^2_{\mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)}).$$

The derivation of such equation from quantum field theory is studied in [7].

3.3 Appendix

Let us begin this section by recalling the disintegration theorem.

3.3.1 The disintegration theorem

Let E, F be separable metric spaces and let $x \in E \mapsto \eta_x \in \mathfrak{P}(F)$ be a measure-valued map. We say that η is a Borel map if $x \in E \mapsto \eta_x(B)$ is a Borel map for any Borel set $B \subset F$. Then we get by the monotone class theorem

$$x \mapsto \int_F f(x,y) d\eta_x(y), \tag{3.3.1}$$

is Borel for every bounded function $f : E \times F \to \mathbb{R}$. Hence by using (3.3.1) the formula

$$\eta(f) = \int_E \int_F f(x,y) d\eta_x(y) d\nu,$$

defines for any $\nu \in \mathfrak{P}(E)$ a unique measure $\eta \in \mathfrak{P}(E \times F)$, that will be denoted $\int_E \eta_x d\nu(x)$. Actually the disintegration theorem below implies for any $\eta \in \mathfrak{P}(E \times F)$ whose first marginal is ν can be represented in this way.

Proposition 3.3.1. *Let E, F be complete separable metric spaces, $\eta \in \mathfrak{P}(E \times F)$, let $\pi : E \times F \to E$ be a borel map and let $\mu^\eta_\pi = \pi_* \eta \in \mathfrak{P}(E)$. Then there exists a $\mu^\eta_\pi - a.e.$ uniquely determined Borel family of probability measures $\{\eta^\pi_x\}_{x\in E} \subset \mathfrak{P}(F)$ such that*

$$\eta = \int_E \eta^\pi_x d\mu^\eta_\pi(x), \tag{3.3.2}$$

i.e

$$\int_{E\times F} f(x,y) d\eta(x,y) = \int_E (\int_F f(x,y) d\eta^\pi_x(y)) d\mu^\eta_\pi(x), \tag{3.3.3}$$

for every Borel map $f : E \times F \to [0, +\infty]$.

3.3.2 Tightness

Denote $\mathcal{Z}_0$ a infinite dimensional separable Hilbert space. Recall that we introduce the space $(\mathcal{Z}_0, d_{w,\mathcal{Z}_0})$ induced by the following distance

$$d_{w,\mathcal{Z}_0}(z_1 - z_2) = \sqrt{\sum_{n\in\mathbb{N}^*} \frac{|\langle z_1 - z_2, e_n\rangle_{\mathcal{Z}_0}|^2}{1+n^2}},\ z_1, z_2 \in \mathcal{Z}_0.$$

One of the main arguments in the proof of Theorem 3.1.1 is the weak tightness of a family of measures (see the weak tightness step in the proof). Therefore we recall below the definition and the criterions used in this proof. We say that a set $\mathcal{K} \subset \mathfrak{P}(\mathcal{Z}_0)$ is *tight* if,

$$\forall \lambda > 0, \exists K_\lambda \text{ compact in } (\mathcal{Z}_0, \|.\|_{\mathcal{Z}_0}) \text{ such that } |\mu(\mathcal{Z}_0 \backslash K_\lambda)| \leqslant \lambda, \qquad \textbf{(Tightness)}$$

and *weakly tight* if,

$$\forall \lambda > 0, \exists K_\lambda \text{ compact in } (\mathcal{Z}_0, d_{w,\mathcal{Z}_0}) \text{ such that } |\mu(\mathcal{Z}_0 \backslash K_\lambda)| \leqslant \lambda \qquad \textbf{(Weak tightness)}$$

A useful characterisation is given here (for more details see Chapter V in [5]). The tightness (resp. weak tightness) condition for a subspace $\mathcal{K}$ is equivalent to an integral condition, i.e there exists a function $\varphi : \mathcal{Z}_0 \to [0, +\infty]$, whose sublevels $\{x \in \mathcal{Z}_0 / \ \varphi(x) \leqslant c\}$ are compact in $(\mathcal{Z}_0, \|.\|_{\mathcal{Z}_0})$ (resp. $(\mathcal{Z}_0, d_{w,\mathcal{Z}_0})$), such that

$$\sup_{\mu \in \mathcal{K}} \int_{\mathcal{Z}_0} \varphi(x) d|\mu|(x) < +\infty.$$

We also use the following tightness criterion:
Let X, X_1, X_2 be separable metric spaces and let $r^i : X \to X_i$ be continuous maps such that the product map

$$r := r^1 \times r^2 : X \to X_1 \times X_2$$

is proper. Let $\mathcal{K} \subset \mathfrak{P}(X)$ be such that $\mathcal{K}_i := r^i_*(\mathcal{K})$ is tight in $\mathfrak{P}(X)$ for $i = 1, 2$. Then also $\mathcal{K}$ is tight in $\mathfrak{P}(X)$.

3.3.3 Projection in finite dimension

Theorem 3.1.1 is obtained in a infinite dimensional separable Hilbert space $\mathcal{Z}_0$. The proof is based on the projection $\pi^d, \hat{\pi}^d, \pi^{d,T}$ introduced in the diagram (3.1) and definitions (3.1.8)-(3.1.9) of the projected velocity vector fields v_t^d and $\hat{v}_t{}^d$. Then the following Lemma is fundamental in our approach.

Lemma 3.3.2. *Let $t \mapsto \mu_t : [0, T] \mapsto \mathfrak{P}(\mathcal{Z}_0)$ be a weakly narrowly continuous map on $\mathcal{Z}_0$ such that the triple $(\mu_t, v_t, \mathcal{Z}_0)$ is solution of the* **weak Liouville equation** . *Then the following assertions hold true.*
i) *The triples $(\mu_t^d, v_t^d, \mathcal{Z}_0)$ and $(\hat{\mu}_t^d, \hat{v}_t^d, \mathcal{Z}_0)$ are solutions of the* **weak Liouville equations**.
ii) *The velocity field v_t^d and $\hat{v}_t^d$ satisfy the following inequality*

$$\|v_t^d\|_{L^1(\mathbb{R}^d, d\mu_t^d)} = \|\hat{v}_t{}^d\|_{L^1(\mathcal{Z}_0, d\hat{\mu}_t^d)} \leqslant \|v_t\|_{L^1(\mathcal{Z}_0, d\mu_t)} < +\infty. \tag{3.3.4}$$

iii) *For every non-decreasing convex function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ the following estimate holds true*

$$\int_0^T \int_{\mathbb{R}^d} \Psi(|v_t^d(x)|) \, d\mu_t^d(x) \, dt \leqslant \int_0^T \int_{\mathcal{Z}_0} \Psi(\|v_t(z)\|_{\mathcal{Z}_0}) \, d\mu_t(z). \tag{3.3.5}$$

Proof. In order to prove the estimates (3.3.4) we introduce a new norm on $\mathbb{R}^d$ given by $\|u\|_{\mathbb{R}^d} = \|\pi^{d,T}u\|_{\mathcal{Z}_0}$ and we compute

$$\begin{aligned}
\|v_t^d\|_{L^1(\mathbb{R}^d,d\mu_t^d)} &= \int_{\mathbb{R}^d} \|\pi^{d,T}\int_{(\pi^d)^{-1}(y)} \pi^d v_t(x) d\mu_{t,y}(x)\|_{\mathbb{R}^d}\, d\mu_t^d(y) \\
&= \int_{\mathbb{R}^d} \|\hat{\pi}^d \int_{(\pi^d)^{-1}(y)} v_t(x)\, d\mu_{t,y}(x)\|_{\mathbb{R}^d}\, d\mu_t^d(y) = \|\hat{v}_t{}^d\|_{L^1(\mathcal{Z}_0,d\hat{\mu}_t^d)} \\
&\leqslant \int_{\mathbb{R}^d}\int_{(\pi^d)^{-1}(y)} \|\hat{\pi}^d v_t(x)\|_{\mathbb{R}^d}\, d\mu_{t,y}(x)\, d\mu_t^d(y) \\
&\leqslant \int_{\mathcal{Z}_0} \|\hat{\pi}^d v_t(x)\|_{\mathcal{Z}_0}\, d\mu_t(x) \leqslant \|v_t\|_{L^1(\mathcal{Z}_0,d\mu_t)} < +\infty.
\end{aligned}$$

The estimate (3.3.5) is a generalization of [89, Lemma 3.9] in the infinite dimensional case by using the disintegration theorem and the Jensen inequality. Consider a regular test function $\varphi = \psi \circ \pi^d$ in (3.1.1), with $\nabla\varphi = (\pi^d)^* \circ \nabla\psi \circ \pi^d$ gives

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{Z}_0} \varphi\, d\mu_t(x) &= \int_{\mathcal{Z}_0} \langle \pi^d(v_t), \nabla\psi \circ \pi^d\rangle\, d\mu_t(x) \\
&= \int_{\mathbb{R}^d}\int_{(\pi^d)^{-1}(y)} \langle \pi^d(v_t), \nabla\psi\circ\pi^d\rangle\, d\mu_{t,y}(x)\, d\mu_t^d(y) \\
\Longrightarrow \frac{d}{dt}\int_{\mathbb{R}^d} \psi\, d\mu_t^d(x) &= \int_{\mathbb{R}^d} \langle v_t^d(y), \nabla\psi(y)\rangle\, d\mu_t^d(y).
\end{aligned}$$

Therefore for $t \in [0,T]$, the triple $(\mu_t^d, v_t^d, \mathbb{R}^d)$ is a solution of the **weak Liouville equation**. It follows similarly that the triple $(\hat{\mu}_t, \hat{v}_t{}^d, \mathcal{Z}_0)$ is also a solution of the **weak Liouville equation**. ∎

Chapter 4

Mean field theory: The multiparticle interaction

4.1 Introduction

In the present Chapter, we consider the mean field problem for a system of many quantum particles described by a N-body Schrödinger Hamiltonian which is typically a sum of a kinetic energy and a multi-particle interaction. As we mentionned in the introduction, there are several approaches to the derivation of the mean field limit. However, most of these approaches are concerned by the propagation of some particular states, namely coherent or factorized states. Moreover, a superposition of coherent (or factorized) vectors can not be handled as a consequence of the previous mentioned results. This limitation motivated the extension of the mean field approximation to a fairly general class of quantum states. The mean field approximation relies on the Wigner measures. One result in [11] deals with a bounded multi-particle interaction and assumes a certain compactness property on quantum states, called the condition (**PI**). This assumption guaranties that all reduced density matrices converge in the trace-norm towards the expected limit (see Subsection 2.2.3). In some sense, the condition (**PI**) avoids any possible defect of compactness in the prepared states. It is therefore significant to extend the mean field approximation to a general class of prepared states which may present this defect of compactness phenomena (i.e. the condition (**PI**) is not required). The second result [12] has been discussed in the introduction (see Theorem 1.1.5). Although the strategy of the latter work is different and seems more powerful than the one employed in [11], the result of [12] does not completely surpass the one obtained in [11] because of the type of assumption on the potential. For example, the main result [12, Theorem 1.1] or Theorem 1.1.5 apply only for bounded potentials which have some decay at infinity. Moreover, the assumption (A3) in [12] (W is a form compact perturbation of the Laplacian) is not even stable by constant perturbation which is otherwise a harmless translation with no effect on the evolution of states. Furthermore, the proof of the main result in [12] relies heavily in the particular structure of the many-body Schrödinger operator. In the present article, we complement the picture in the case of many-body bounded interaction and extend the mean field propagation result in [11] to a more general class of quantum states which may present a defect of compactness phenomena. However, we know from the work [12] that this is not possible unless we strength the assumption on the interaction. Therefore we assume that the many-particle interaction is given by compact operators (see (4.1.1) and (2)) then a general propagation result holds true (see Theorem 4.1.2). Besides the fact that our assumption (2) is mathematically rather natural, it is also motivated by the numerical simulations developed by the Pawilowski B. in [93] where he used discrete models of many-body Hamiltonians with an interaction which is in fact compact. To prove our main result, namely Theorem 4.1.2, we follow the same strategy as [12]. While following the same strategy as [12], we emphasize the key points, simplify the presentation and improve the technical arguments by working with an abstract many-body operator (4.1.1) and no more relying on the particular structure of the Schrödinger operator. In our case the free Hamiltonian is generated by a non negative operator A and the interaction by some compact multi-particle $\tilde{Q}_\ell$ operators (4.1.1). And actually, our main result does not overlap with the one in [12] since in the latter work the two-body interaction is a multiplication operator and hence can not be compact even when it is bounded.

We work in the bosonic Fock space

$$\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \bigvee^{n} \mathcal{Z} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} \,,$$

modelled on a one particle separable complex Hilbert space $\mathcal{Z}$, where $\mathcal{S}_n$ is the symmetrization projection defined on $\mathcal{Z}^{\otimes n}$ by

$$\mathcal{S}_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(n)} \,,$$

with the sum running over all permutations of n elements. If not specified tensor products and orthogonal direct sums are considered in their Hilbert completed version. We are interested in the mean field dynamics of the many-body Hamiltonian with multi-particle interaction

$$H_\varepsilon^{(n)} = H_\varepsilon^{0,(n)} + \sum_{\ell=2}^{r} \varepsilon^\ell \frac{n!}{(n-\ell)!} \mathcal{S}_n(\tilde{Q}_\ell \otimes \mathrm{Id}_{\bigvee^{n-\ell}\mathcal{Z}}) \mathcal{S}_n \,, \quad n \geqslant 2r \,, \tag{4.1.1}$$

in the asymptotic regime $\varepsilon \to 0$, $n\varepsilon \to 1$. Here the $\tilde{Q}_\ell$'s are bounded symmetric operators on $\bigvee^\ell \mathcal{Z}$ and

$$H_\varepsilon^{0,(n)} = \varepsilon \sum_{i=1}^{n} \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \underbrace{A}_{i} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \,, \tag{4.1.2}$$

where A is a given self-adjoint operator. Within the second quantization (see Section 2.1.3), the operator $H_\varepsilon^{(n)}$ (resp. $H_\varepsilon^{0,(n)}$) can be written as a restriction to the subspace $\bigvee^n \mathcal{Z}$ of the operator H_ε (resp. H_ε^0) defined on the Fock space and given by:

$$H_\varepsilon = H_\varepsilon^0 + Q^{Wick} \quad , \quad Q(z) = \sum_{\ell=2}^{r} \langle z^{\otimes \ell} \,, \, \tilde{Q}_\ell z^{\otimes \ell} \rangle, \tag{4.1.3}$$

$$H_\varepsilon^0 = d\Gamma(A) = \langle z \,, \, Az \rangle^{Wick} \,. \tag{4.1.4}$$

The mean field energy functional is

$$h(z, \bar{z}) = \langle z, Az \rangle + \ Q(z) \,, \tag{4.1.5}$$

so that the mean field dynamics are given by the non linear equation

$$i\partial_t z_t = \partial_{\bar{z}} h(z_t, \bar{z}_t) = A z_t + \partial_{\bar{z}} Q(z_t). \tag{4.1.6}$$

In our framework, the annihilation and creation operators, $a(z_1)$ and $a^*(z_2)$, with z_1, z_2 in $\mathcal{Z}$, satisfy the ε-dependent Canonical Commutation Relations (CCR):

$$[a(z_1), a^*(z_2)] = \varepsilon \langle z_1 \,,\, z_2 \rangle \ \mathrm{Id} \ .$$

Recall that the Weyl operator, for $\xi \in \mathcal{Z}$, is defined by

$$\mathcal{W}(\xi) = e^{i \frac{a(\xi) + a^*(\xi)}{\sqrt{2}}},$$

and the number operator $\mathbf{N}$ is

$$\mathbf{N} = \mathrm{d}\Gamma(\ \mathrm{Id}\ )\,.$$

We refer the reader to Section 2.1.2 for a brief review of the second quantization and these related operators.

Our approach in the derivation of the mean field dynamics uses Wigner measures. For reader convenience, we recall the definition below.

Definition 4.1.1. *Let $\mathcal{E}$ be an infinite subset of $(0, +\infty)$ such that $0 \in \overline{\mathcal{E}}$. Let $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$ ($\varrho_\varepsilon \geqslant 0$ and $\mathrm{Tr}\ [\varrho_\varepsilon] = 1$) such that:*

$$\exists \delta > 0\,, \exists C_\delta > 0\,, \forall \varepsilon \in \mathcal{E}\,, \qquad \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta < \infty\,.$$

The set $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$ of Wigner measures associated with $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ is the set of Borel probability measures on $\mathcal{Z}$, μ, such that there exists an infinite subset $\mathcal{E}' \subset \mathcal{E}$ with $0 \in \overline{\mathcal{E}'}$ and

$$\forall \xi \in \mathcal{Z}\,, \quad \lim_{\mathcal{E}' \ni \varepsilon \to 0} \mathrm{Tr}\ \left[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\right] = \int_{\mathcal{Z}} e^{2i\pi \,\mathrm{Re}\, \langle \xi, z \rangle} d\mu(z)\,.$$

Our main result will be proved under the following assumptions:

A1. *The operator A with the domain $D(A)$ is self-adjoint in $\mathcal{Z}$.*

A2. *For all $\ell \in \{2, \ldots, r\}$, the operator $\tilde{Q}_\ell$ is compact and self-adjoint in $\bigvee^\ell \mathcal{Z}$.*

The notations **A1-A2** introduced above are the only assumptions used in this Chapter. We will prove the following result.

Theorem 4.1.2. *Let $(\varrho_\varepsilon)_{\varepsilon \in (0, \bar{\varepsilon})}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$ such that*

$$\exists \delta > 0\,, \exists C_\delta > 0\,, \forall \varepsilon \in (0, \bar{\varepsilon})\,, \qquad \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta < \infty\,, \tag{4.1.7}$$

and which admits a unique Wigner measure μ_0. Under the Assumptions **(A1)**-**(A2)**, *the family $(e^{-i\frac{t}{\varepsilon}H_\varepsilon} \varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon \in (0, \bar{\varepsilon})}$ admits for every $t \in \mathbb{R}$ a unique Wigner measure μ_t, which is the push-forward $\Phi(t, 0)_* \mu_0$ of the initial measure μ_0 by the flow associated with*

$$\begin{cases} i\partial_t z_t = A z_t + \partial_{\bar{z}} Q(z_t), \\ z_{t=0} = z_0. \end{cases} \tag{4.1.8}$$

As we have mentioned previously, many authors used the BBGKY hierarchy method to justify the mean field limit for particular states which are coherent or factorized (see e.g. [47, 110]). Actually, the coherent and factorized states verify the condition (**PI**). Thus, our result on propagation of Wigner measures is more general and indeed when the initial state verifies the condition (**PI**), even without being particular, the convergence of reduced density matrices holds true. This was already proved in [12, Theorem 4.1] and we avoid the repetition of this result since our main contribution concerns the propagation of states with a defect of compactness (i.e. The condition (**PI**) is not satisfied).

The proof of Theorem 4.1.2 requires few steps. The operator H_ε with a suitable domain is proved to be self-adjoint in Proposition 4.2.1. Proposition 4.2.3 ensures that the Cauchy problem (4.1.8) defines a global flow on $\mathscr{Z}$.
Beside this, the proof consists in several steps that we briefly sketch here. For the first four points of this proof below we consider more regular states ϱ_ε.

1. By setting

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,, \tag{4.1.9}$$

$$\text{and}\qquad \tilde{\varrho}_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_\varepsilon^0}\varrho_\varepsilon(t)e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}\,, \tag{4.1.10}$$

   we write

$$\begin{aligned}\mathrm{Tr}\,[\tilde{\varrho}_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)] = {} & \mathrm{Tr}\,[\varrho_\varepsilon\mathcal{W}(\sqrt{2}\pi\xi)] \\ & + i\int_0^t \mathrm{Tr}\,[\tilde{\varrho}_\varepsilon(s)\mathcal{W}(\sqrt{2}\pi\xi)\sum_{j=1}^r \varepsilon^{j-1}\mathcal{O}_j(s,\xi)]ds\,,\end{aligned} \tag{4.1.11}$$

   where the $\mathcal{O}_j(s,\xi)$'s are Wick quantized observables which satisfy some uniform estimates.

2. The number estimates given in Proposition (**Number estimate**) provide equicontinuity properties of the quantity $\mathrm{Tr}\,\left[\tilde{\varrho}_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)\right]$ w.r.t $(\xi,t)\in\mathbb{R}\times\mathscr{Z}$. So that a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to 0 can be extracted such that for all times $t\in\mathbb{R}$,

$$\mathcal{M}(\tilde{\varrho}_\varepsilon(t),\varepsilon\in\mathcal{E}) = \{\tilde{\mu}_t\}$$

   with $\mathcal{E}=\{\varepsilon_k\,,k\in\mathbb{N}\}$.

3. With the number estimates, we get rid of the terms for $j\geqslant 2$ as $\varepsilon\to 0$ in (4.1.11). The compactness Assumption **(A2)** is used in Proposition 4.3.1 when we take the limit in all the remaining terms of (4.1.11) for general initial data $\tilde{\varrho}_\varepsilon$. Subsequently, the measure $\tilde{\mu}_t$ is a weak solution of the Liouville equation

$$i\partial_t\tilde{\mu}_t + \{Q_t(z),\tilde{\mu}_t\} = 0\,, \tag{4.1.12}$$

   with $Q_t(z) = Q(e^{-itA}z)$.

4. Finally, we follow the same lines as in [12] and refer to measure transportation tools developed in [5] in order to prove $(e^{-itA})_*\tilde{\mu}_t = \Phi(t,0)_*\mu_0$, and hence we get

$$\mathcal{M}(\varrho_\varepsilon(t),\varepsilon\in(0,\bar{\varepsilon})) = \{\mu_t\}.$$

5. The last point of this proof is a truncation scheme used in [12] for more general states.

4.2 Quantum and mean-field dynamics

In this section we show that the quantum and the classical dynamics are both well defined for all times.

4.2.1 Self-adjoint realization

Proposition 4.2.1.
(i) For any $n \in \mathbb{N}$, the operator $H_\varepsilon^{(n)}$ given by (4.1.1) *with domain $D(\mathrm{d}\Gamma(A)) \cap \bigvee^n \mathcal{Z}$ is a self-adjoint operator in $\bigvee^n \mathcal{Z}$.*
(ii) The operator H_ε, given by (4.1.3)*, is self-adjoint in $\Gamma_s(\mathcal{Z})$ with the domain defined by*

$$(\Psi \in D(H_\varepsilon)) \Leftrightarrow \left(\begin{array}{l} \Psi \in \Gamma_s(\mathcal{Z}), \\ \forall n \in \mathbb{N}\,,\ \Psi^{(n)} \in D(H_\varepsilon^{(n)})\,, \\ \sum_{n=0}^{\infty} \|H_\varepsilon^{(n)} \Psi^{(n)}\|^2 < +\infty \end{array} \right).$$

Proof. (i) For $n \in \mathbb{N}$ and according to (4.1.3)-(4.1.4) the operator $H_\varepsilon^{(n)}$ equals

$$H_\varepsilon^{(n)} = H_\varepsilon^{0,(n)} + V_\varepsilon^{(n)}, \tag{4.2.1}$$

with $V_\varepsilon^{(n)} = \sum_{\ell=2}^r \{Q_\ell(z)\}^{Wick}_{|\vee^n \mathcal{Z}}$ with $Q_\ell(z) = \langle z^{\otimes \ell}, \tilde{Q}_\ell z^{\otimes \ell}\rangle$.
For $\phi^{(n)} \in D(d\Gamma(A)) \cap \bigvee^n \mathcal{Z}$, a simple computation gives

$$\begin{aligned} \|V_\varepsilon^{(n)} \phi^{(n)}\| &\leqslant \sum_{\ell=2}^r \varepsilon^\ell \frac{n!}{(n-\ell)!} \|\mathcal{S}_n(\tilde{Q}_\ell \otimes \ \mathrm{Id}_{\ \bigvee^{n-\ell} \mathcal{Z}})\phi^{(n)}\|, \\ &\leqslant \sum_{\ell=2}^r \varepsilon^\ell \frac{n!}{(n-\ell)!} \|\tilde{Q}_\ell\| \ \|\phi^{(n)}\|_{\bigvee^n \mathcal{Z}}, \\ &\leqslant C_{r,\varepsilon,n} \|\phi^{(n)}\|_{\bigvee^n \mathcal{Z}}. \end{aligned}$$

So $V_\varepsilon^{(n)}$ is a bounded self-adjoint perturbation of $H_\varepsilon^{0,(n)}$ and therefore $H_\varepsilon^{(n)}$ is self-adjoint on $D(H_\varepsilon^{0,(n)})$.
(ii) Proposition [12, A1] is applied here, with $A_n = H_\varepsilon^{0,(n)} + V_\varepsilon^{(n)}$ yields the self-adjointness of H_ε . ∎

Once we have defined the quantum dynamics, we can then write an integral formula giving the propagation of normal states. However, instead of considering

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon} \varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon},$$

we will rather work with

$$\tilde{\varrho}_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_\varepsilon^0} \varrho_\varepsilon(t) e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}.$$

With the convention of Section 2.1.3, $\mathrm{D}^j b$ denotes the j-th differential of b with respect to $(z, \bar{z})$:

$$\mathrm{D}^j[b(z)][\xi] = \sum_{|\alpha|+|\beta|=j} \frac{j!}{\alpha!\beta!} \langle \xi^{\otimes \beta}\ ,\ \partial_z^\alpha \partial_{\bar{z}}^\beta b(z) \xi^{\otimes \alpha}\rangle.$$

Proposition 4.2.2. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$. Assume that*

$$\forall k \in \mathbb{N},\ \exists C_k > 0\ ,\forall \varepsilon \in (0,\bar\varepsilon),\ \ \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^k] \leqslant C_k. \tag{4.2.2}$$

Then for all $\xi \in \mathcal{Z}$, the function $t \mapsto \mathrm{Tr}\,[\tilde\varrho_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)]$ belongs to $C^1(\mathbb{R})$ and the following formula holds:

$$\mathrm{Tr}\,[\tilde\varrho_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)] = \mathrm{Tr}\,[\varrho_\varepsilon\mathcal{W}(\sqrt{2}\pi\xi)] + \tag{4.2.3}$$

$$i\int_0^t \mathrm{Tr}\,[\tilde\varrho_\varepsilon(s)\mathcal{W}(\sqrt{2}\pi\xi)\{\sum_{j=1}^r \varepsilon^{j-1}\frac{(i\pi)^j}{j!}\mathrm{D}^j[Q(e^{-isA}z)][\xi]\}^{Wick}]ds.$$

Proof. We denote $\langle\mathbf{N}\rangle = (1+\mathbf{N}^2)^{\frac12}$.
The quantity $\mathrm{Tr}\,[(\tilde\varrho_\varepsilon(t)-\tilde\varrho_\varepsilon(s))\mathcal{W}(\sqrt{2}\pi\xi)]$ is actually equal to

$$\mathrm{Tr}\,[\varrho_\varepsilon\langle\mathbf{N}\rangle^r(e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0} - e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0})\langle\mathbf{N}\rangle^{-r}\mathcal{W}(\sqrt{2}\pi\xi)e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon}] \tag{4.2.4}$$

$$+\ \mathrm{Tr}\,[\varrho_\varepsilon\langle\mathbf{N}\rangle^r e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}\langle\mathbf{N}\rangle^{-r}\mathcal{W}(\sqrt{2}\pi\xi)\langle\mathbf{N}\rangle^r(e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon} - e^{i\frac{s}{\varepsilon}H_\varepsilon^0}e^{-i\frac{s}{\varepsilon}H_\varepsilon})\langle\mathbf{N}\rangle^{-r}]. \tag{4.2.5}$$

By differentiating first for $u \in D(H_\varepsilon^0) \cap \bigvee^n \mathcal{Z}$ and then extending the result by continuity, we get for $u \in \bigvee^n \mathcal{Z}$:

$$e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}u = e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}u + \frac{i}{\varepsilon}\int_t^s e^{i\frac{\sigma}{\varepsilon}H_\varepsilon}Q^{Wick}e^{-i\frac{\sigma}{\varepsilon}H_\varepsilon^0}u\ d\sigma\,.$$

The number estimate in Proposition (**Number estimate**) combined with

$$e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0} = \bigoplus_{n=0}^\infty e^{i\frac{s}{\varepsilon}H_\varepsilon^{(n)}}e^{-i\frac{s}{\varepsilon}H_\varepsilon^{0,(n)}},$$

implies for all $u \in \Gamma_s(\mathcal{Z})$:

$$\langle\mathbf{N}\rangle^{-r}e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}u = \langle\mathbf{N}\rangle^{-r}e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}u + \frac{i}{\varepsilon}\int_t^s e^{i\frac{\sigma}{\varepsilon}H_\varepsilon}\langle\mathbf{N}\rangle^{-r}Q^{Wick}e^{-i\frac{\sigma}{\varepsilon}H_\varepsilon^0}u\ d\sigma\,.$$

The integrand is continuous in $\Gamma_s(\mathcal{Z})$ w.r.t σ for any $u \in \Gamma_s(\mathcal{Z})$. Taking the limit as $s \to t$ leads to

$$s-\lim_{s\to t}\frac{1}{t-s}\langle\mathbf{N}\rangle^{-r}(e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0} - e^{i\frac{s}{\varepsilon}H_\varepsilon}e^{-i\frac{s}{\varepsilon}H_\varepsilon^0}) = \frac{i}{\varepsilon}\langle\mathbf{N}\rangle^{-r}e^{i\frac{t}{\varepsilon}H_\varepsilon}Q^{Wick}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}. \tag{4.2.6}$$

Similarly (by exchanging H_ε^0 and H_ε) we get:

$$s-\lim_{s\to t}\frac{1}{t-s}\langle\mathbf{N}\rangle^{-r}(e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon} - e^{i\frac{s}{\varepsilon}H_\varepsilon^0}e^{-i\frac{s}{\varepsilon}H_\varepsilon}) = -\frac{i}{\varepsilon}\langle\mathbf{N}\rangle^{-r}e^{i\frac{t}{\varepsilon}H_\varepsilon^0}Q^{Wick}e^{-i\frac{t}{\varepsilon}H_\varepsilon}. \tag{4.2.7}$$

Notice that

$$\langle\mathbf{N}\rangle^{-r}e^{i\frac{t}{\varepsilon}H_\varepsilon}Q^{Wick}e^{-i\frac{t}{\varepsilon}H_\varepsilon^0} \in \mathcal{L}(\Gamma_s(\mathcal{Z})), \qquad \mathrm{Tr}\,[\varrho_\varepsilon\langle\mathbf{N}\rangle^r] < C_r < +\infty,$$

and

$$\mathcal{W}(\sqrt{2}\pi\xi)e^{i\frac{t}{\varepsilon}H_\varepsilon^0}e^{-i\frac{t}{\varepsilon}H_\varepsilon} \in \mathcal{L}(\Gamma_s(\mathcal{Z})).$$

Thus the trace (4.2.4) divided by $t - s$ is well defined and converges as $s \to t$ thanks to (4.2.6). In equation (4.2.7), remark that

$$\langle \mathbf{N} \rangle^{-r} e^{i\frac{t}{\varepsilon}H_\varepsilon^0} Q^{Wick} e^{-i\frac{t}{\varepsilon}H_\varepsilon} \in \mathcal{L}(\Gamma_s(\mathcal{Z})),$$

and

$$\forall \xi \in \mathcal{Z},\ \langle \mathbf{N} \rangle^{-r} \mathcal{W}(\sqrt{2}\pi\xi) \langle \mathbf{N} \rangle^{r} \in \mathcal{L}(\Gamma_s(\mathcal{Z})),$$

owing to the Lemma 6.2 in [6]. Since for all $u \in \Gamma_s(\mathcal{Z})$

$$s \mapsto e^{i\frac{s}{\varepsilon}H_\varepsilon} e^{-i\frac{s}{\varepsilon}H_\varepsilon^0} u \in \mathcal{C}(\mathbb{R}, \Gamma_s(\mathcal{Z})),$$

the trace (4.2.5) divided by $t - s$ is well defined and converges as $s \to t$ thanks to (4.2.7). Therefore the following integral formula holds true, with the help of (2.1.12),

$$\begin{aligned}\mathrm{Tr}\,[\tilde{\varrho}_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)] = {} & \mathrm{Tr}\,[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)] + \\ & \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}\,[\tilde{\varrho}_\varepsilon(s)[Q_s^{Wick}\mathcal{W}(\sqrt{2}\pi\xi) - \mathcal{W}(\sqrt{2}\pi\xi)Q_s^{Wick}]]ds, \end{aligned} \tag{4.2.8}$$

with $Q_s(z) = Q(e^{-isA}z)$. We conclude by using Lemma 2.1.28. ∎

4.2.2 The nonlinear (Hartree) equation

In this section we shall prove the global well posedness of the mean field dynamics. So we consider the Cauchy problem in $\mathcal{Z}$:

$$\begin{cases} i\partial_t z_t = A z_t + \partial_{\bar{z}} Q(z_t), \\ z_{t=0} = z_0. \end{cases} \tag{4.2.9}$$

Proposition 4.2.3. *Under the assumptions* **(A1)** *and* **(A2)***, for all $z_0 \in \mathcal{Z}$, the previous Cauchy problem admits a unique mild solution z_t in $\mathcal{C}^0(\mathbb{R}, \mathcal{Z}) \cap \mathcal{C}^1(\mathbb{R}, D(A)')$.*
Furthermore the Cauchy problem:

$$\begin{cases} \partial_t \tilde{z}_t = v(t, \tilde{z}_t) = -ie^{itA}[\partial_{\bar{z}} Q](e^{-itA}\tilde{z}_t), \\ \tilde{z}_{t=0} = z_0. \end{cases} \tag{4.2.10}$$

is equivalent with the initial problem and admits a unique solution $\tilde{z}_t \in \mathcal{C}^1(\mathbb{R}, \mathcal{Z})$.
Furthermore this equation implies:

- $\forall t \in \mathbb{R},\ \|z_t\|_{\mathcal{Z}} = \|z_0\|_{\mathcal{Z}} = \|\tilde{z}_t\|_{\mathcal{Z}}$,
- *The velocity field $v(t, z) = -ie^{itA}[\partial_{\bar{z}} Q](e^{-itA}z)$, satisfies*

$$\forall t \in \mathbb{R},\ \|v(t, z)\|_{\mathcal{Z}} \leqslant r\ M(\sum_{j=2}^{r} \|z\|_{\mathcal{Z}}^{2j-1}), \tag{4.2.11}$$

with $M = \max_{j=2,\ldots,r} \|\tilde{Q}_j\|$.

Proof. It is enough to consider only positive times $t > 0$. We will prove that $z \to v(t,z) = -ie^{itA}[\partial_{\bar z}Q](e^{-itA}z)$ is locally Lipschitz in $\mathcal{Z}$ which will give the local existence and uniqueness on a time interval $[0,T^*[$ for the equation (4.2.10). Then we can recover solutions of the original equation (4.2.9) by setting $z_t = e^{-itA}\tilde z_t$.
Let z, y be in $\mathcal{Z}$,

$$\begin{aligned}\|v(t,z)-v(t,y)\|_{\mathcal{Z}} &\leqslant \|[\partial_{\bar z}Q](e^{-itA}z)-[\partial_{\bar z}Q](e^{-itA}y)\|,\\ &\leqslant \sum_{j=2}^{r} j|\,(\langle z_t^{\otimes j-1}|\vee \text{ Id }_{\mathcal{Z}})\tilde Q_j(z_t^{\otimes j})-\langle y_t^{\otimes j-1}|\vee \text{ Id }_{\mathcal{Z}})\tilde Q_j(y_t^{\otimes j})\,|\,.\end{aligned}$$

Thus, by setting $M = \max_{j=2,\ldots,r}\|\tilde Q_j\|$, for all $z, y \in B(0,R)$, there exists a non negative constant $C_R > 0$ such that:

$$\|v(t,z)-v(t,y)\|_{\mathcal{Z}} \leqslant r\ C_R\ M\|z-y\|_{\mathcal{Z}}.$$

Thus the Cauchy-Lipschitz theorem gives a unique solution $\tilde z_t$ in $\mathcal{C}^1([0,T^*[,\mathcal{Z})$. The previous calculus with $y=0$ gives, for $t \in [0,T^*[$, the estimate:

$$\|v(t,z)\|_{\mathcal{Z}} \leqslant M\ r\sum_{j=2}^{r}\|z\|_{\mathcal{Z}}^{2j-1}.$$

It remains to prove $\|z_t\| = \|z_0\|$ for all $t \in [0,T^*[$ which ensures that $T^* = +\infty$. In fact

$$\begin{aligned}\partial_t\|\tilde z_t\|^2 = 2\,\mathrm{Re}\,\langle \tilde z_t, \partial_t \tilde z_t\rangle &= -2\,\mathrm{Re}\,\langle \tilde z_t, ie^{itA}[\partial_{\bar z}Q](e^{-itA}\tilde z_t)\rangle\\ &= -2\,\mathrm{Re}\ \ i\langle e^{-itA}\tilde z_t, [\partial_{\bar z}Q](e^{-itA}\tilde z_t)\rangle\\ &= -2\sum_{\ell=2}^{r}\mathrm{Re}\ \ i\langle e^{-itA}\tilde z_t, [\partial_{\bar z}Q_\ell](e^{-itA}\tilde z_t)\rangle\\ &= -2\sum_{\ell=2}^{r}\mathrm{Re}\ \ [i\ell\,\underbrace{Q_\ell(e^{-itA}\tilde z_t)}_{\in\mathbb{R}}] = 0.\end{aligned}$$

So this shows that $\|\tilde z_t\|_{\mathcal{Z}} = \|z_t\|_{\mathcal{Z}} = \|z_0\|_{\mathcal{Z}}$ and the mass conservation is proved. By setting $z_t = e^{-itA}\tilde z_t$ and using the fact that the solution $\tilde z_t$ satisfies

$$\tilde z_t = z_0 - i\int_0^t e^{isA}[\partial_{\bar z}Q](e^{-isA}\tilde z_s)ds,$$

we obtain

$$z_t = e^{-itA}z_0 - i\int_0^t e^{i(s-t)A}[\partial_{\bar z}Q](z_s)ds.$$

Hence the function $t \mapsto z_t$ belongs to $\mathcal{C}^0(\mathbb{R},\mathcal{Z}) \cap \mathcal{C}^1(\mathbb{R}, D(A)')$ and it is a mild solution of (4.2.9). ∎

4.3 Propagation of Wigner measures

4.3.1 The main convergence arguments

The following proposition will be useful in the derivation of the transport equation. It is mainly due to the compactness of the $\tilde{Q}_j$'s.

Proposition 4.3.1. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$. Assume the Assumptions* **(A1)** *and* **(A2)** *are satisfied and*

$$\forall k \in \mathbb{N},\ \exists C_k > 0,\ \forall \varepsilon \in (0, \bar{\varepsilon}),\ \ \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^k] \leqslant C_k.$$

Assume furthermore that:

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0, \bar{\varepsilon})) = \{\mu\}.$$

Then for all $\xi \in \mathcal{Z}$ and all $t \in \mathbb{R}$:

$$\lim_{\varepsilon\longrightarrow 0} \mathrm{Tr}\,[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\{\mathrm{D}[Q(e^{-itA}z)][\xi]\}^{Wick}] = \int_{\mathcal{Z}} e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}\mathrm{D}[Q(e^{-itA}z)][\xi]d\mu(z), \tag{4.3.1}$$

where $\mathrm{D}[Q(e^{-itA}z)][\xi] = \langle[\partial_{\bar{z}}Q](e^{-itA}z), e^{-itA}\xi\rangle + \langle e^{-itA}\xi, [\partial_{\bar{z}}Q](e^{-itA}z)\rangle$.

Proof. For $j \in \{2, \ldots r\}$ and $\xi \in \mathcal{Z}$, let $B_j(\xi)$ denote the operator

$$B_j(\xi) = \tilde{Q}_j(\ \mathrm{Id}_{\bigvee^{j-1}\mathcal{Z}} \otimes |\xi\rangle), \tag{4.3.2}$$

and

$$B_j^*(\xi) = (\ \mathrm{Id}_{\bigvee^{j-1}\mathcal{Z}} \otimes \langle\xi|)\tilde{Q}_j.$$

Both operators are compact respectively from $\bigvee^{j-1}\mathcal{Z}$ to $\bigvee^{j}\mathcal{Z}$ and from $\bigvee^{j}\mathcal{Z}$ to $\bigvee^{j-1}\mathcal{Z}$ owing to the assumption **(A2)**. Now, let us check that $\mathrm{D}[Q(e^{-itA}z)][\xi]$ is the sum of symbols with compact kernels. Actually, $Q(e^{-itA}z) = \sum_{j=2}^r \langle z^{\otimes j}, (e^{itA})^{\otimes j}\tilde{Q}_j(e^{-itA})^{\otimes j}z^{\otimes j}\rangle$ with $\tilde{Q}_j^* = \tilde{Q}_j$. In particular with $\overline{Q(z)} = Q(z)$, we obtain

$$\begin{aligned}
\mathrm{D}[Q(e^{-itA}z)][\xi] &= \langle[\partial_{\bar{z}}Q](e^{-itA}z), e^{-itA}\xi\rangle + \langle e^{-itA}\xi, [\partial_{\bar{z}}Q](e^{-itA}z)\rangle, \\
&= \sum_{j=2}^r j[\langle z^{\otimes j}, \tilde{Q}_j(e^{-itA}\xi \vee z^{\otimes j-1})\rangle + \langle\tilde{Q}_j(e^{-itA}\xi \vee z^{\otimes j-1}), z^{\otimes j}\rangle], \\
&= \sum_{j=2}^r j[\langle z^{\otimes j}, B_j(e^{-itA}\xi)z^{\otimes j-1}\rangle + \langle z^{\otimes j-1}, B_j^*(e^{-itA}\xi)z^{\otimes j}\rangle],
\end{aligned}$$

and all the terms involve compact operators. We refer to Lemma 2.2.15 in order to compute the limit of $\mathrm{Tr}\,[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\{\mathrm{D}[Q(e^{-itA}z)][\xi]\}^{Wick}]$ and obtain (4.3.1). ∎

In order to understand the asymptotic behaviour of $\mathrm{Tr}\,[\varrho_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)]$, when ε goes to 0, we shall prove the following proposition.

Proposition 4.3.2. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$. Assume the assumptions* **(A1)** *and* **(A2)** *are satisfied and*

$$\forall k \in \mathbb{N},\ \exists C_k > 0,\ \forall \varepsilon \in (0,\bar\varepsilon),\ \ \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^k] \leqslant C_k.$$

Assume furthermore that

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)) = \{\mu\}.$$

Then for all $\xi \in \mathcal{Z}$ and all $t \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0}\int_0^t \mathrm{Tr}\,[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\{\sum_{j=2}^{r}\varepsilon^{j-1}\frac{(i\pi)^j}{j!}\mathrm{D}^j[Q(e^{-isA}z)][\xi]\}^{Wick}]ds\ = 0.$$

Proof. By using Proposition (**Number estimate**), a simple estimate of the integrand yields for all $s \in [0,t]$

$$\begin{aligned}
&|\ \mathrm{Tr}\,[\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)\{\sum_{j=2}^{r}\varepsilon^{j-1}\frac{(i\pi)^j}{j!}\mathrm{D}^j[Q(e^{-isA}z)][\xi]\}^{Wick}]| \\
&\leqslant C_r\sum_{j=2}^{r}\varepsilon^{j-1}\frac{\pi^j}{j!}\|\langle \mathbf{N}\rangle^{-r}\{\mathrm{D}^j[Q(e^{-isA}z)][\xi]\}^{Wick}\|_{\mathcal{L}(\Gamma_s(\mathcal{Z}))} \\
&\leqslant \sum_{j=2}^{r}\varepsilon^{j-1}\frac{\pi^j}{j!}\tilde C_r\langle\xi\rangle^j,
\end{aligned}$$

with $\langle u\rangle = (1+|u|^2)^{\frac12}$. We conclude therefore by the dominated convergence theorem. ∎

4.3.2 Existence of Wigner measures for all times

Remember the definition of

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon},$$

and

$$\tilde\varrho_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_\varepsilon^0}\varrho_\varepsilon(t)e^{-i\frac{t}{\varepsilon}H_\varepsilon^0}.$$

Proposition 4.3.3. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\Gamma_s(\mathcal{Z})$. Assume the assumptions* **(A1)**-**(A2)** *are satisfied and*

$$\forall k \in \mathbb{N}, \exists C_k > 0,\ \forall \varepsilon \in (0,\bar\varepsilon),\ \ \mathrm{Tr}\,[\varrho_\varepsilon \mathbf{N}^k] \leqslant C_k\,. \tag{4.3.3}$$

For all sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\bar\varepsilon)$ such that $\lim_{n\to+\infty}\varepsilon_n = 0$, there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to+\infty}\varepsilon_{n_k} = 0$ and a family of Borel probability measures $\{\tilde\mu_t, t\in\mathbb{R}\}$ such that:

$$\forall t \in \mathbb{R},\ \mathcal{M}(\tilde\varrho_{\varepsilon_k}(t), k\in\mathbb{N}) = \{\tilde\mu_t\}. \tag{4.3.4}$$

Furthermore,

$$\int_{\mathcal{Z}}\|z\|_{\mathcal{Z}}^{2k}\ d\tilde\mu_t(z) \leqslant C_k,\ \forall k\in\mathbb{N}, \tag{4.3.5}$$

and $\tilde{\mu}_t$ solves the integral equation, for all $\xi \in \mathcal{Z}$,

$$\tilde{\mu}_t(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}) = \tilde{\mu}_0(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}) - \pi \int_0^t \tilde{\mu}_s(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle} \mathrm{D}[Q(e^{-isA}.)][\xi])ds,$$

$$= \tilde{\mu}_0(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}) + i \int_0^t \tilde{\mu}_s(\{\tilde{Q}_s\ ,\ e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}(z)\})ds, \tag{4.3.6}$$

by setting for $b_1,b_2 \in \mathcal{P}_{p,q}(\mathcal{Z})$

$$\{b_1, b_2\}(z) = \partial_z b_1(z).\partial_{\bar{z}} b_2(z) - \partial_z b_2(z).\partial_{\bar{z}} b_1(z).$$

Proof. The extraction of such subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and the existence of a family of Borel probability measures $\tilde{\mu}_t$ have been proved in [12] by a diagonal extraction process relying on some Ascoli type argument. We skip the proof of this step since the result in [12] applies to our case without modification. Let p_n be the projection on $\mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$ with $(e_i)_{i\in\mathbb{N}}$ an ONB of $\mathcal{Z}$. Since $d\Gamma(p_n) \leqslant \mathbf{N}$, it follows that

$$\int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^{2k} d\tilde{\mu}_t(z) = \sup_{n\in\mathbb{N}} \int_{\mathcal{Z}} \|p_n z\|_{\mathcal{Z}}^{2k} d\tilde{\mu}_t(z) = \sup_{n\in\mathbb{N}} \{\liminf_{\varepsilon\to 0}\ \mathrm{Tr}\ [\tilde{\varrho}_\varepsilon(t)(d\Gamma(p_n))^k]\}$$
$$\leqslant \liminf_{\varepsilon\to 0}\ \mathrm{Tr}\ [\varrho_\varepsilon \mathbf{N}^k] \leqslant C_k\ .$$

This proves (4.3.5). For the derivation of the integral equation (4.3.6), we have according to (4.2.3),

$$\mathrm{Tr}\ [\tilde{\varrho}_\varepsilon(t)\mathcal{W}(\sqrt{2}\pi\xi)] = \ \mathrm{Tr}\ [\varrho_\varepsilon \mathcal{W}(\sqrt{2}\pi\xi)] +$$

$$i \int_0^t \mathrm{Tr}\ [\tilde{\varrho}_\varepsilon(s)\mathcal{W}(\sqrt{2}\pi\xi)\{\sum_{j=1}^r \varepsilon^{j-1}\frac{(i\pi)^j}{j!}\mathrm{D}^j[Q(e^{-isA}z)][\xi]\}^{Wick}]ds.$$

The estimate of Proposition 4.3.2 implies that all the terms $j = 2, ..., r$ of the sum in the right side go to 0 as $\varepsilon \to 0$. For the last term, we use Proposition 4.3.1 for $\tilde{\varrho}_\varepsilon(s)$ thanks to the fact that

$$\mathrm{Tr}\ [\tilde{\varrho}_\varepsilon(t)\mathbf{N}^k] = \ \mathrm{Tr}\ [\varrho_\varepsilon \mathbf{N}^k] \leqslant C_k\,.$$

Thus taking the limit as $\varepsilon \to 0$ yields

$$\tilde{\mu}_t(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}) = \tilde{\mu}_0(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}) - \pi \int_0^t \tilde{\mu}_s(e^{2i\pi \operatorname{Re}\langle \xi,.\rangle} \mathrm{D}[Q(e^{-isA}.)][\xi])ds.$$

We conclude with

$$\begin{aligned} i\{\tilde{Q}_s\,,\, e^{2i\pi \operatorname{Re}\langle \xi,.\rangle}\}(z) &= i(\langle [\partial_{\bar{z}} Q](e^{-isA}z)\,,\, \partial_{\bar{z}} e^{2i\pi \operatorname{Re}\langle \xi,z\rangle}\rangle - \langle \partial_{\bar{z}} e^{2i\pi \operatorname{Re}\langle \xi,z\rangle}\,,\, [\partial_{\bar{z}} Q](e^{-isA}z)\rangle), \\ &= i\pi(\langle [\partial_{\bar{z}} Q](e^{-isA}z)\,,\, i\xi\rangle - \langle i\xi, [\partial_{\bar{z}} Q](e^{-isA}z)\rangle)\, e^{2i\pi \operatorname{Re}\langle \xi,z\rangle}\,, \\ &= -\pi \mathrm{D}[Q(e^{-isA}z)][\xi]\, e^{2i\pi \operatorname{Re}\langle \xi,z\rangle}\,. \end{aligned}$$

∎

4.3.3 The Liouville equation fulfilled by the Wigner measures.

The previous integral equation (4.3.6) can be interpreted as a continuity equation, in the infinite dimensional Hilbert space $\mathcal{Z}$, fulfilled by the Wigner measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$.
We introduce some classes of cylindrical functions on $\mathcal{Z}$. Denote $\mathbb{P}$ the space of the finite rank orthogonal projections on $\mathcal{Z}$. A function is in the cylindrical Schwarz space $\mathcal{S}_{cyl}(\mathcal{Z})$ (resp. $C^\infty_{0,cyl}(\mathcal{Z})$) if

$$\exists \mathfrak{p} \in \mathbb{P},\ \exists g \in \mathcal{S}(\mathfrak{p}\mathcal{Z})(\text{resp. } C^\infty_{0,cyl}(\mathfrak{p}\mathcal{Z})),\ \forall z \in \mathcal{Z}, f(z) = g(\mathfrak{p}z).$$

The space $C^\infty_{0,cyl}(\mathbb{R}\times\mathcal{Z})$ which enforces the compact support in the first variable, will be useful too. Denote $L_{\mathfrak{p}}(dz)$ the Lebesgue measure associated with the finite dimensional subspace $\mathfrak{p}\mathcal{Z}$. The Fourier transform is given on $\mathcal{S}_{cyl}(\mathcal{Z})$ by :

$$\mathcal{F}[f](\xi) = \int_{\mathfrak{p}\mathcal{Z}} f(z) e^{-2i\pi \operatorname{Re}\langle z,\xi\rangle_{\mathcal{Z}}} L_{\mathfrak{p}}(dz),$$

$$f(z) = \int_{\mathfrak{p}\mathcal{Z}} \mathcal{F}[f](\xi) e^{2i\pi \operatorname{Re}\langle z,\xi\rangle_{\mathcal{Z}}} L_{\mathfrak{p}}(d\xi).$$

Then call $Prob_2(\mathcal{Z})$ the set of Borel probability measures μ finite second moment, i.e. $\int_{\mathcal{Z}} \|z\|^2_{\mathcal{Z}}\, d\mu(z) < \infty$. On this space the Wasserstein distance is given by the formula:

$$W_2(\mu_1,\mu_2) = \sqrt{\inf_{\mu\in\Gamma(\mu_1,\mu_2)} \int_{\mathcal{Z}} \|z_1 - z_2\|^2_{\mathcal{Z}} d\mu(z_1,z_2)}, \tag{4.3.7}$$

with $\Gamma(\mu_1,\mu_2)$ the set of probability measures μ on $\mathcal{Z}\times\mathcal{Z}$ such that the marginals $(\Pi_1)_*\mu = \mu_1$ and $(\Pi_2)_*\mu = \mu_2$. Let $\beta(\mathcal{Z})$ be the family of all Borel probability measures on a Hilbert space $\mathcal{Z}$. Here $\Pi_j, j = 1,2$, are the canonical projections on the first and the second component respectively.
From now, after introducing a Hilbert basis $(e_n)_{n\in\mathbb{N}^*}$, the space $\mathcal{Z}$ can be equipped with the distance

$$d_w(x_1 - x_2) = \sqrt{\sum_{n\in\mathbb{N}} \frac{|\langle x_1 - x_2, e_n\rangle|^2}{1+n^2}}.$$

It induces a topology globally weaker than the weak topology. However these topology coincide on bounded sets of $\mathcal{Z}$.
The norm and d_w topology give rise two distinct notions of narrow convergence of probability measures. On the one hand, a sequence $(\mu_n)_{n\in\mathbb{N}}$ is narrowly convergent to $\mu \in \beta(\mathcal{Z})$ if

$$\lim_{n\to+\infty} \int_{\mathcal{Z}} f(z) d\mu_n(z) = \int_{\mathcal{Z}} f(z) d\mu(z), \tag{4.3.8}$$

for every function $f \in \mathcal{C}^0_b(\mathcal{Z}, \|.\|_{\mathcal{Z}})$, the space of continuous and bounded real functions defined on $\mathcal{Z}$ with the norm topology. On the other hand, a sequence $(\mu_n)_{n\in\mathbb{N}}$ is weakly narrowly convergent if the limit (5.6.1) holds for all $f \in \mathcal{C}^0_b(\mathcal{Z}, d_w)$. Our sequences of probability measures are assumed to have a uniformly bounded moment $\int_{\mathcal{Z}} \|z\|^{2k}_{\mathcal{Z}} d\mu_n(z) \leqslant C_k$ for some $k \geqslant 1$. Within this framework, the narrow convergence is equivalent to the convergence with respect to the Wasserstein distance W_2 in $Prob_2(\mathcal{Z})$ according to Proposition 7.1.5 in [5]. With the same moment condition, the weak narrow convergence is equivalent to the convergence (4.3.8) for all $f \in \mathcal{S}_{cyl}(\mathcal{Z})$ or for all $f \in \mathcal{C}^\infty_{0,cyl}(\mathcal{Z})$, acccording to Lemma 5.1.12 f) in [5].

Proposition 4.3.4. *Assume that the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies:*

$$\forall\alpha\in\mathbb{N},\ \exists\, C_\alpha>0\,,\forall\varepsilon\in(0,\bar\varepsilon),\ \ \mathrm{Tr}\left[\varrho_\varepsilon\mathbf{N}^\alpha\right]\leqslant C_\alpha. \tag{4.3.9}$$

Consider a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty}\varepsilon_k=0$,

$$\mathcal{M}(\tilde\varrho_{\varepsilon_k}(t),k\in\mathbb{N})=\{\tilde\mu_t\}. \tag{4.3.10}$$

Then the probability measure $\tilde\mu_t$ defined on $\mathcal{Z}$ satisfies

1. *When $(e_n)_{n\in\mathbb{N}^*}$ a Hilbert basis of $\mathcal{Z}$ and $\mathcal{Z}$ is endowed with the distance $d_w(z_1,z_2)=\sqrt{\sum_{n\in\mathbb{N}^*}\frac{|<z_1-z_2,e_n>|}{n^2}}$, the measure $\tilde\mu_t$ is weakly narrowly continuous with respect to t.*
2. *This is a weak solution to the (continuity) Liouville equation*

$$\partial_t\tilde\mu_t+i\{Q_t,\tilde\mu_t\}=0, \tag{4.3.11}$$

in the sense that for all $f\in C^\infty_{0,cyl}(\mathbb{R}\times\mathcal{Z})$

$$\int_{\mathbb{R}}\int_{\mathcal{Z}}\left(\partial_t f+i\{Q_t,f\}\right)d\tilde\mu_t(z)dt=0, \tag{4.3.12}$$

with $Q_t(z)=Q(e^{-itA}z)$.

Proof. **a)** The characteristic function G of the measure $\tilde\mu_t$ is given by

$$G(\eta,t)=\tilde\mu_t(e^{-2i\pi\,\mathrm{Re}\,\langle\eta,z\rangle})\,.$$

The following inequality holds:

$$|G(\eta,t)-G(\eta',t)|\leqslant 2\pi\|\eta-\eta'\|_{\mathcal{Z}}\int_{\mathcal{Z}}\|z\|_{\mathcal{Z}}d\tilde\mu_t(z). \tag{4.3.13}$$

Since the uniform estimate $\int_{\mathcal{Z}}1+\|z\|^2_{\mathcal{Z}}d\tilde\mu_t(z)\leqslant C_2$ is true for all times, we get for all η,η' in $\mathcal{Z}$ and for $t\in\mathbb{R}$,

$$|G(\eta,t)-G(\eta',t)|\leqslant\pi\|\eta-\eta'\|C_2. \tag{4.3.14}$$

b) According to Proposition 4.3.3 and (4.3.6),

$$\tilde\mu_{t'}(e^{2i\pi\,\mathrm{Re}\,\langle\xi,.\rangle})-\tilde\mu_t(e^{2i\pi\,\mathrm{Re}\,\langle\xi,.\rangle})=-\pi\int_t^{t'}\tilde\mu_s(e^{2i\pi\,\mathrm{Re}\,\langle\xi,.\rangle}\mathrm{D}[Q(e^{-isA}.)][\xi])ds.$$

We use the estimate (4.2.11) and get

$$|\mathrm{D}[Q(e^{-isA}z)][\xi]|\leqslant 2\|e^{-isA}\xi\|_{\mathcal{Z}}\ |[\partial_{\bar z}Q](e^{-isA}z)|\leqslant 2\|\xi\|Mr\sum_{j=2}^{r}\|z\|^{2j-1}_{\mathcal{Z}}.$$

Thus for $\xi \in \mathcal{Z}$

$$\begin{aligned}|G(\xi,t')-G(\xi,t)| &\leqslant \Big|\pi\int_t^{t'} G(\xi,s)\mathrm{D}[Q(e^{-isA}z)][\xi]ds\Big|,\\ &\leqslant 2\pi|t-t'|\|\xi\|_{\mathcal{Z}}Mr\sup_{s\in[t,t']}\sum_{j=2}^{r}\int_{\mathcal{Z}}\|z\|_{\mathcal{Z}}^{2j-1}d\tilde{\mu}_s(z),\\ &\leqslant C|t-t'|\|\xi\|_{\mathcal{Z}},\end{aligned}$$

since $\|z\|_{\mathcal{Z}}^{2j-1}\leqslant\frac{1}{2}(1+\|z\|_{\mathcal{Z}}^{2(2j-1)})\in L^1(\mathcal{Z},\tilde{\mu}_t)$ and

$$\sum_{j=2}^{r}\int_{\mathcal{Z}}\|z\|_{\mathcal{Z}}^{2j-1}d\tilde{\mu}_s(z)\leqslant C_r,$$

with a time independent constant C_r. Hence for all ξ in $\mathcal{Z}$ and for all t,t' in $\mathbb{R}$:

$$|G(\xi,t')-G(\xi,t)|\leqslant 2\pi C_rMr|t-t'|\|\xi\|_{\mathcal{Z}}. \tag{4.3.15}$$

1. Take now $g\in\mathcal{S}_{cyl}(\mathcal{Z})$ based on $\mathfrak{p}\mathcal{Z}$ and the equality holds:

$$I_g(t)=\int_{\mathcal{Z}}g(z)d\tilde{\mu}_t(z)=\int_{\mathfrak{p}\mathcal{Z}}\mathcal{F}[g](\eta)G(\eta,t)dL_{\mathfrak{p}}(\eta). \tag{4.3.16}$$

We shall establish the continuity of I_g on $\mathbb{R}$. Indeed

- $t\longrightarrow\mathcal{F}[g](\eta)G(\eta,t)$ is continue owing to (4.3.15)
- $\eta\longrightarrow\mathcal{F}[g](\eta)G(\eta,t)$ is bounded by a $L_{\mathfrak{p}}(d\eta)$-integrable function thanks to (4.3.13) and $\mathcal{F}[g]\in\mathcal{S}(\mathfrak{p}\mathcal{Z})$.

Thus we have the continuity of I_g for all $g\in\mathcal{S}_{cyl}(\mathcal{Z})$. Furthermore the uniform estimate condition

$$\forall\alpha\in\mathbb{N},\ \int_{\mathcal{Z}}\|z\|_{\mathcal{Z}}^{2\alpha}d\tilde{\mu}_t(z)\leqslant C_\alpha,$$

with C_α time independent allow us to apply lemma 5.1.12-f) in [5] and to assert that the map $t\to\tilde{\mu}_t$ is weakly narrowly continuous.

2. We integrate the expression (4.3.6) with respect to $\mathcal{F}[g](\eta)L_{\mathfrak{p}}(dz)$:

$$\forall t\in\mathbb{R},\ \forall g\in\mathcal{S}_{cyl}(\mathcal{Z}),\int_{\mathcal{Z}}g(z)d\tilde{\mu}_t(z)=\int_{\mathcal{Z}}g(z)d\tilde{\mu}_0(z)+i\int_0^t\int_{\mathcal{Z}}\{Q_s,g\}d\tilde{\mu}_s(z)ds.$$

Hence I_g belongs to $C^1(\mathbb{R})$ and satisfies:

$$\partial_tI_g(t)=i\int_{\mathcal{Z}}\{Q_t,g\}(z)d\tilde{\mu}_t(z).$$

Multiplying this expression by a function $\phi \in C_0^\infty(\mathbb{R})$ and integrating by parts lead to

$$\int_{\mathbb{R}} \partial_t I_g(t)\phi(t)dt = i\int_{\mathbb{R}\times\mathcal{Z}} \{Q_t, g\}(z)d\tilde{\mu}_t(z)\phi(t)dt.$$

Integrating by parts gives

$$\int_{\mathbb{R}\times\mathcal{Z}} g(z)d\tilde{\mu}_t(z)\phi'(t)dt + i\int_{\mathbb{R}\times\mathcal{Z}} \{Q_t, g\}\phi(t)\, d\tilde{\mu}_t(z)dt = 0,$$

$$\text{or} \int_{\mathbb{R}\times\mathcal{Z}} (\partial_t f(t,z) + i\{Q_t, f\})\, d\tilde{\mu}_t(z)dt = 0,$$

with $f(t,z) = g(z)\phi(t)$.

We conclude by using the density of $C_0^\infty(\mathbb{R}) \otimes^{alg} C_{0,cyl}^\infty(\mathcal{Z})$ in $C_{0,cyl}^\infty(\mathbb{R}\times\mathcal{Z})$.

∎

4.3.4 Convergence toward the mean field dynamics

Proposition 4.3.5. *Assume that the family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ on $\Gamma_s(\mathcal{Z})$ fulfills the assumptions* **(A1)**-**(A2)***, with the uniform control*

$$\forall\alpha \in \mathbb{N},\, \exists C_\alpha > 0\,, \forall\varepsilon \in (0,\bar{\varepsilon}),\;\; \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}^\alpha\right] \leqslant C_\alpha,$$

and

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu_0\}.$$

Then for any time $t \in \mathbb{R}$ the family $(\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar{\varepsilon})}$ admits a unique Wigner measure μ_t equal to $\Phi(t,0)_*\mu_0$, where Φ is the flow associated with the well defined Hartree equation owing to Proposition 4.2.3. Moreover, the map $t \mapsto \mu_t \in Prob_2(\mathcal{Z})$ is continuous with respect to the Wasserstein distance W_2.

Proof. Take $\tilde{\varrho}_\varepsilon(t)$ as in (4.1.10) and consider the Hartree equation (4.1.6) with the flow $\Phi(t,s)$ corresponding to:

$$i\partial_t z_t = \partial_{\bar{z}} h(z,\bar{z}) = Az_t + \partial_{\bar{z}} Q(z_t), \tag{4.3.17}$$

on $\mathcal{Z}$ and the flow $\tilde{\Phi}(t,s)$ associated with

$$\partial_t \tilde{z}_t = v(t,\tilde{z}_t) \quad \text{with} \quad v(t,z) = -ie^{itA}[\partial_{\bar{z}} Q](e^{-itA}z)\,.$$

Proposition 4.2.3 provides the following estimate

$$\|v(t,z)\|_{\mathcal{Z}} \leqslant Mr \sum_{j=2}^{r} \|z\|_{\mathcal{Z}}^{2j-1}\,,$$

with $M = \max_{j=2,\dots,r} \|\tilde{Q}_j\|$. Recall that $(\tilde{\mu}_t)$ are the Wigner measures defined for all times and associated with a subsequence $(\tilde{\varrho}_{\varepsilon_{n_k}})_{k\in\mathbb{N}}$, hence we obtain

$$\begin{aligned}\|v(t,z)\|_{L^2(\mathcal{Z},\tilde{\mu}_t)} &= \sqrt{\int_{\mathcal{Z}} \|v(t,z)\|^2_{\mathcal{Z}}\, d\tilde{\mu}_t(z)} \\ &\leqslant Mr\sqrt{\sum_{j=2}^{r}\int_{\mathcal{Z}} \|z\|^{2(2j-1)}_{\mathcal{Z}}\, d\tilde{\mu}_t(z)} \in L^1([-T,T]).\end{aligned}$$

This holds since

$$\forall j \in \mathbb{N},\ \int_{\mathcal{Z}} \|z\|^{2j}_{\mathcal{Z}}\, d\tilde{\mu}_t(z) \leqslant C_j\,,$$

with C_j time independent.
Now, using Proposition 5.6.2 the measure $\tilde{\mu}_t$ satisfies

$$\partial_t\tilde{\mu}_t + i\{Q_t,\tilde{\mu}_t\} = \partial_t\tilde{\mu}_t + \nabla^T(v(t,z)\tilde{\mu}_t) = 0,$$

in the weak sense and the map $t \mapsto \tilde{\mu}_t \in Prob_2(\mathcal{Z})$ is weakly narrowly continuous. Moreover, the velocity field $v(t,.)$ satisfies the condition $\|v(t,z)\|_{L^2(\mathcal{Z},\mu_t)}$ belongs to $L^1([-T,T])$.
Thus, $\tilde{\mu}_t$ verifies the conditions of [12, Proposition C1] with $I = [-T,T]$ and then $\tilde{\mu}_t$ is continuous with respect to the Wasserstein distance W_2. So now the measures $\tilde{\mu}_t$ satisfy all the hypotheses of [12, Proposition C4], i.e.:

- $t \mapsto \tilde{\mu}_t \in Prob_2(\mathcal{Z})$ is W_2-continuous.
- For all $T > 0$, $|v(t,z)|_{L^2(\mathcal{Z},\mu_t)}$ belongs to $L^1([-T,T])$.
- $\tilde{\mu}_t$ is the weak solution to:
$$\partial_t\tilde{\mu}_t + \nabla^T(v(t,z)\tilde{\mu}_t) = 0,$$

subsequently $\tilde{\mu}_t = \tilde{\phi}(t,0)_*\mu_0$ and

$$\mathcal{M}(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})) = \{\tilde{\mu}_t\},$$

for any time $t \in \mathbb{R}$. By noticing that

$$\varrho_\varepsilon(t) = e^{-it\frac{t}{\varepsilon}H^0_\varepsilon}\varrho_\varepsilon e^{it\frac{t}{\varepsilon}H^0_\varepsilon},$$

we get

$$\mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})) = \{\mu_t\},$$

which finish the proof of Theorem 4.1.2 for regular data.

∎

4.3.5 Evolution of the Wigner measure for general data

In this subsection, we shall prove Theorem 4.1.2 for general data. We used the same truncation scheme used in [12]. Hence consider a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfying the assumption of Theorem 4.1.2, i.e.:

$$\exists\delta > 0\,, \exists C_\delta > 0\,, \forall\varepsilon \in \mathcal{E}\,, \qquad \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}^\delta\right] \leqslant C_\delta < \infty\,.$$

There exists another family $(\varrho_\varepsilon^{(m)})_{m\in\mathbb{N}}$ such that $\mathrm{Tr}\left[\varrho_\varepsilon^{(m)}\right] = 1$,

$$\forall k \in \mathbb{N}\,, \exists C_k > 0\,, \forall\varepsilon \in (0,\bar{\varepsilon})\,, \qquad \mathrm{Tr}\left[\varrho_\varepsilon^{(m)}\mathbf{N}^k\right] \leqslant C_k < \infty$$

and

$$\lim_{m\to+\infty}\sup_{\varepsilon\in(0,\bar{\varepsilon})}\|\varrho_\varepsilon - \varrho_\varepsilon^{(m)}\|_{\mathcal{L}^1} = 0. \tag{4.3.18}$$

Indeed by setting

$$\varrho_\varepsilon^{(m)} = \frac{1}{\mathrm{Tr}\left[\chi_m(\mathbf{N})\varrho_\varepsilon\chi_m(\mathbf{N})\right]}\chi_m(\mathbf{N})\varrho_\varepsilon\chi_m(\mathbf{N}),$$

with $\chi_m(n) = \chi(\frac{n}{m})$ and $0 \leqslant \chi \leqslant 1, \chi \in C_0^\infty(\mathbb{R})$ and $\chi \equiv 1$ in a neighborhood of 0, the result (4.3.18) holds.

The family $(\varrho_\varepsilon^{(m)})_{m\in\mathbb{N}}$ is satisfying the assumption of Proposition 4.3.5 and then by extracting a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that for all $t \in \mathbb{R}$

$$\mathcal{M}(\varrho_{\varepsilon_{n_k}}^{(m)}(t), k \in \mathbb{N}) = \{\phi(t,0)_*\mu_0^{(m)}\},$$

with $\phi(t,0)$ the flow of the Hartree equation (4.1.6), and $\mu_0^{(m)}$ the Wigner measure associated with $\varrho_\varepsilon^{(m)}$. Hence by setting $\mu_t \in \mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon}))$, there exists a subsequence $(\varepsilon_l)_{l\in\mathbb{N}}$ such that

$$\mathcal{M}(\varrho_{\varepsilon_l}(t), l \in \mathbb{N}) = \{\mu_t\}.$$

Then a computation of the total variation by the triangle inequality gives

$$\begin{aligned}\int_{\mathcal{Z}}|\mu_t - \phi(t,0)_*\mu_0| &\leqslant \int_{\mathcal{Z}}|\mu_t - \phi(t,0)_*\mu_0^{(m)}| + \int_{\mathcal{Z}}|\phi(t,0)_*\mu_0^{(m)} - \phi(t,0)_*\mu_0| \\ &\leqslant \int_{\mathcal{Z}}|\mu_t - \phi(t,0)_*\mu_0^{(m)}| + \int_{\mathcal{Z}}|\mu_0^{(m)} - \mu_0|.\end{aligned}$$

By taking the limit when $m \to +\infty$ we get $\int_{\mathcal{Z}}|\mu_t - \phi(t,0)_*\mu_0| = 0$, hence

$$\mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})) = \{\phi(t,0)_*\mu_0\}.$$

4.4 Examples

In this Section we will give some examples of states which do not satisfy the condition (**PI**). So that the results in [10, 11] can not be applied on these examples however they satisfy the assumptions of our Theorem 4.1.2. We recall the assertion (**PI**) below.

Definition 4.4.1. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states such that*

$$\forall\alpha\in\mathbb{N},\ \exists C_\alpha>0\,,\forall\varepsilon\in(0,\bar\varepsilon),\ \ \mathrm{Tr}\,[\varrho_\varepsilon\mathbf{N}^\alpha]\leqslant C_\alpha, \tag{4.4.1}$$

and

$$\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))=\{\mu\}.$$

Then we say that $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies the condition (**PI**) *if:*

$$\lim_{\varepsilon\to0}\ \mathrm{Tr}\,[\varrho_\varepsilon\mathbf{N}^k]=\int_{\mathcal{Z}}\|z\|_{\mathcal{Z}}^{2k}\,d\mu(z),\ \ \forall k\in\mathbb{N}\,. \tag{4.4.2}$$

Let us consider two kinds of normal states on the Fock space $\Gamma_s(\mathcal{Z})$, namely the coherent and Hermite states.

- The coherent states are given by

$$\varrho_\varepsilon(f)=|E(f)\rangle\langle E(f)|=|\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}f)\Omega\rangle\,\langle\mathcal{W}(\frac{\sqrt{2}}{i\varepsilon}f)\Omega|\,, \tag{4.4.3}$$

  where $f\in\mathcal{Z}$ and Ω is the vacuum vector of the Fock space.

- The Hermite states are given by

$$\varrho_\varepsilon(f)=|f^{\otimes k}\rangle\langle f^{\otimes k}|\,,\quad k=[\frac{1}{\varepsilon}],\quad f\in\mathcal{Z}. \tag{4.4.4}$$

For some coherent states or Hermite states the property (**PI**) was proved in [10] by a simple computation

$$\lim_{\varepsilon\to0}\ \mathrm{Tr}\,[\varrho_\varepsilon\mathbf{N}^k]=\|f\|_{\mathcal{Z}}^{2k}=\delta_f(\|z\|_{\mathcal{Z}}^{2k}). \tag{4.4.5}$$

Thus there is no loss of compactness for those states when f does not depend on ε. However, for our examples we will consider coherent and Hermite states where the vector f is ε-dependent. More precisely, let $(f_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of vectors in $\mathcal{Z}$ such that :

A3. $\|f_\varepsilon\|_{\mathcal{Z}}=1$ *and f_ε weakly convergent to f_0 but not strongly, (i.e. $\|f_0\|_{\mathcal{Z}}<1$).*

Then we shall prove that the family of normal states $\varrho_\varepsilon(f_\varepsilon)$ given by (4.4.3) or (4.4.4) provides a good example of states which does not satisfy the property (**PI**) but only the uniform condition (4.4.1). The assumption (3) is motivated by the situation where each particle of the system is not in a pure state but in a superposition, for example $f_\varepsilon=\frac{e_N+e_1}{\sqrt{2}}$ where $N=[\frac{1}{\varepsilon}]$ and $\{e_n\}_{n\in\mathbb{N}}$ is an O.N.B with e_1 and e_N representing the ground and the excited states respectively of a single quantum particle.

We recall two useful Propositions from [9]. Consider two families of vectors $(u_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ and $(v_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ in $\Gamma_s(\mathcal{Z})$. With the family of trace class operators,

$$\varrho_\varepsilon^{(u_\varepsilon,v_\varepsilon)}=|u_\varepsilon\rangle\langle v_\varepsilon|,$$

complex-valued Wigner measures can be defined by a simple linear decomposition, specified in [9]-Proposition 6.4. Recall that for a family of Hermite states $u_\varepsilon=u^{\otimes[\frac{1}{\varepsilon}]}$,

$$\mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon,u_\varepsilon)})=\{\delta_u^{S^1}\}=\{\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}u}d\theta\}.$$

Proposition 4.4.2. *Assume that the family $(u_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ and $(v_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ of vectors in the Fock space satisfy the uniform estimates*

$$\|(1+\mathbf{N})^{\frac{\delta}{2}}u_\varepsilon\| + \|(1+\mathbf{N})^{\frac{\delta}{2}}v_\varepsilon\| \leqslant C,\ \|u_\varepsilon\| = \|v_\varepsilon\| = 1\,,$$

for some $\delta > 0$ and $C > 0$. If additionally any $\mu \in \mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon,u_\varepsilon)})$ and any $\nu \in \mathcal{M}(\varrho_\varepsilon^{(v_\varepsilon,v_\varepsilon)})$ are mutually orthogonal, then

$$\mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon,v_\varepsilon)}, \varepsilon \in (0,\bar\varepsilon)) = \{0\}.$$

This is equivalent to

$$\lim_{\varepsilon\to 0}\langle u_\varepsilon, b^{Weyl} v_\varepsilon\rangle = 0,$$

for any $b \in \mathcal{S}_{cyl}(\mathcal{Z})$.

Proposition 4.4.3. *Assume the same assumptions as in the Proposition above with the additional condition $\mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon,u_\varepsilon)}) = \{\mu\}$ and $\mathcal{M}(\varrho_\varepsilon^{(v_\varepsilon,v_\varepsilon)}) = \{\nu\}$. Then the family of trace class operators $(\varrho_\varepsilon^{(u_\varepsilon+v_\varepsilon,u_\varepsilon+v_\varepsilon)})_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies*

$$\mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon+v_\varepsilon,u_\varepsilon+v_\varepsilon)}) = \{\mu+\nu\}.$$

Corollary 4.4.4. *Let $\varrho_\varepsilon(f_\varepsilon)$ be either the family of coherent states* (4.4.3) *or Hermite states* (4.4.4) *with f_ε satisfying* **(A3)**. *Then the condition* **(PI)** *fails for $\varrho_\varepsilon(f_\varepsilon)$. However, we get for all $t > 0$, $\mathcal{M}(\varrho_\varepsilon(f_\varepsilon)(t), \varepsilon \in (0,\bar\varepsilon)) = \{\mu_t\}$ with $\mu_t = \delta_{\Phi(t,0)f_0}$ in the case of coherent states and $\mu_t = \delta^{S^1}_{\Phi(t,0)f_0}$ in the case of Hermite states, where $\Phi(t,0)$ is the flow associated with*

$$i\partial_t z_t = Az_t + \partial_{\bar z}Q(z_t).$$

Furthermore, set $u_\varepsilon = u^{\otimes[\frac{1}{\varepsilon}]}$, $u \in \mathcal{Z}$, then for all $t \in \mathbb{R}$

$$\mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon+E(f_\varepsilon),u_\varepsilon+E(f_\varepsilon))}, \varepsilon \in (0,\bar\varepsilon)) = \{\delta^{S^1}_{u_t} + \delta_{f_t}\}\,,$$

with $u_t = \Phi(t,0)u$ and $f_t = \Phi(t,0)f_0$.

Proof. The proof splits on several steps.

- *Identification of the Wigner measure:*
  <u>For coherent states</u>:
  In order to identify the Wigner measure associated with $\varrho_\varepsilon(f_\varepsilon)$, at time $t = 0$, use the well known formula
  $$\mathrm{Tr}\ (\varrho_\varepsilon(f_\varepsilon)\mathcal{W}(\sqrt{2}\pi\xi)) = e^{2i\pi\,\mathrm{Re}\,\langle\xi,f_\varepsilon\rangle}e^{-\varepsilon\frac{\|\sqrt{2}\pi\xi\|^2}{4}}\,,$$
  here the right hand side converges to $e^{2i\pi\,\mathrm{Re}\,\langle\xi,f_0\rangle} = \mathcal{F}^{-1}(\delta_{f_0})$ when ε goes to 0. Therefore $\mathcal{M}(\varrho_\varepsilon(f_\varepsilon), \varepsilon \in (0,\bar\varepsilon)) = \{\delta_{f_0}\}$.

  <u>For Hermite states</u>:
  A simple computation yields for any $b \in \mathcal{P}^\infty_{p,q}(\mathcal{Z})$,
  $$\lim_{\varepsilon\to 0}\ \mathrm{Tr}\ [\varrho_\varepsilon b^{Wick}] = \lim_{\varepsilon\to 0}\langle f_\varepsilon^{\otimes n}, b^{Wick} f_\varepsilon^{\otimes n}\rangle = \langle f_0^{\otimes q}, \tilde b f_0^{\otimes p}\rangle = \int_{\mathcal{Z}} b(z)\,d\delta^{S^1}_{f_0}(z).$$

Then by applying Proposition 6.15 in [9], one proves that $\mathcal{M}(\varrho_\varepsilon(f_\varepsilon), \varepsilon \in (0, \bar{\varepsilon})) = \{\delta_{f_0}^{S^1}\}$.
For the superposition of orthogonal states:
Recall that $\varrho_\varepsilon^{(u+E(f_\varepsilon), u+E(f_\varepsilon))} = |u + E(f_\varepsilon)\rangle\langle u + E(f_\varepsilon)|$ and according to Proposition 4.4.3 and 4.4.2

$$\begin{aligned}\mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon+E(f_\varepsilon), u_\varepsilon+E(f_\varepsilon))}, \varepsilon \in (0, \bar{\varepsilon})) &= \mathcal{M}(\varrho_\varepsilon^{(u_\varepsilon, u_\varepsilon)}) + \mathcal{M}(\varrho_\varepsilon^{(E(f_\varepsilon), E(f_\varepsilon))}) + \{0\} \\ &= \{\delta_u^{S^1} + \delta_{f_0}\}.\end{aligned}$$

- *Uniform estimates:*
Let $k \in \mathbb{N}$, the following uniform estimate holds
For the coherent states:

$$\mathrm{Tr}\ (\varrho_\varepsilon(f_\varepsilon)\mathbf{N}^k) = \langle \Omega, \mathcal{W}^*(\frac{\sqrt{2}}{i\varepsilon} f_\varepsilon)\mathbf{N}^k \mathcal{W}(\frac{\sqrt{2}}{i\varepsilon} f_\varepsilon)\Omega\rangle \leqslant C_k \|(\mathbf{N}+1)^{k/2}\Omega\|^2 \leqslant C_k.$$

For Hermite states:
In this case $\varrho_\varepsilon(f_\varepsilon) = |f_\varepsilon^{\otimes N}\rangle\langle f_\varepsilon^{\otimes N}|$ with $N = [\frac{1}{\varepsilon}]$ is the number of particles.
Notice that for all $k \in \mathbb{N}$

$$\mathrm{Tr}\ [\varrho_\varepsilon \mathbf{N}^k] = (\varepsilon n)^k \|f_\varepsilon\|_{\mathcal{Z}}^2 = (\varepsilon n)^k. \tag{4.4.6}$$

For the superposition of orthogonal states:
In this case $\varrho_\varepsilon^{(u_\varepsilon+E(f_\varepsilon), u_\varepsilon+E(f_\varepsilon))} = |u_\varepsilon + E(f_\varepsilon)\rangle\langle u_\varepsilon + E(f_\varepsilon)|$,

$$\forall k \in \mathbb{N},\ \ \mathrm{Tr}\ [(|u_\varepsilon + E(f_\varepsilon)\rangle\langle u_\varepsilon + E(f_\varepsilon)|)\mathbf{N}^k] \leqslant C_k.$$

- *The condition fails* (**PI**)*:*
For coherent states:
A simple computation of $\mathrm{Tr}\ (\varrho_\varepsilon(f_\varepsilon)\mathbf{N}^k)$ when $k = 1$ gives the following equality

$$\mathrm{Tr}\ (\varrho_\varepsilon(f_\varepsilon)\mathbf{N}) = \langle E(f_\varepsilon), (\|z\|_{\mathcal{Z}}^2)^{Wick} E(f_\varepsilon)\rangle = \|f_\varepsilon\|_{\mathcal{Z}}^2 = 1,$$

and

$$\int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^2 d\delta_{f_0}(z) = \|f_0\|_{\mathcal{Z}}^2 .$$

But $\|f_\varepsilon\|_{\mathcal{Z}}^2$ does not converge to $\|f_0\|_{\mathcal{Z}}^2$ because f_ε does not converge strongly to f_0. Hence the quantity $\mathrm{Tr}\ (\varrho_\varepsilon(f_\varepsilon)\mathbf{N})$ does not converge to $\int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^2\, d\delta_{f_0}(z)$. Then the condition (**PI**) is not satisfied.
For the Hermite states:
It is easy to see that the condition (**PI**) fails. Indeed on the one hand $\lim_{\varepsilon\to 0}\ \mathrm{Tr}\ [\varrho_\varepsilon \mathbf{N}^k] = 1$ but on the other hand $\int_{\mathcal{Z}} \|z\|_{\mathcal{Z}}^{2k}\, d\delta_{f_0} = \|f_0\|_{\mathcal{Z}}^{2k} < 1$.

For the superposition of orthogonal states:
Assume that the family $(\varrho_\varepsilon^{(u_\varepsilon+E(f_\varepsilon), u_\varepsilon+E(f_\varepsilon))})_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies the condition (**PI**). Fix $k = 1$ and

compute

$$\mathrm{Tr}\left[\varrho_\varepsilon^{(u_\varepsilon+E(f_\varepsilon),u_\varepsilon+E(f_\varepsilon))}\mathbf{N}^k\right] = \quad \langle u_\varepsilon, \mathbf{N}u_\varepsilon\rangle + \langle E(f_\varepsilon), \mathbf{N}E(f_\varepsilon)\rangle + \langle u_\varepsilon, \mathbf{N}E(f_\varepsilon)\rangle \\ +\langle E(f_\varepsilon), \mathbf{N}u_\varepsilon\rangle$$

The two last terms converge to 0 when $\varepsilon \to 0$ since

$$\begin{aligned}\lim_{\varepsilon\to 0,\, \varepsilon n\to 1} |\langle E(f_\varepsilon), \mathbf{N}u_\varepsilon\rangle| &= \lim_{\varepsilon\to 0,\, \varepsilon n\to 1} |\langle u_\varepsilon, \mathbf{N}E(f_\varepsilon)\rangle| \\ &= \lim_{\varepsilon\to 0,\, \varepsilon n\to 1} \varepsilon n |\langle u_\varepsilon, \frac{\varepsilon^{-\frac{n}{2}} e^{-\frac{1}{2\varepsilon}}}{\sqrt{n!}} f_\varepsilon\rangle| = 0,\end{aligned}$$

according to formula

$$E(f_\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^{-\frac{n}{2}} e^{-\frac{\|f_\varepsilon\|^2}{2\varepsilon}}}{\sqrt{n!}} f_\varepsilon^{\otimes n}.$$

Besides, the family of Hermite states $(\varrho_\varepsilon^{(u_\varepsilon,u_\varepsilon)})_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies the condition (**PI**) for $k = 1$, hence the family $(\varrho_\varepsilon^{(E(f_\varepsilon),E(f_\varepsilon))})_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies the condition (**PI**) for $k = 1$ which is wrong.

- *Propagation*:
  All the examples (coherent, Hermite and orthogonal states) satisfy the assumptions of Theorem 4.1.2. Hence, for all $t > 0$
  $$\mathcal{M}(\varrho_\varepsilon(f_\varepsilon)(t), \varepsilon \in (0,\bar\varepsilon)) = \{\mu_t\},$$
  with $\mu_t = \Phi(t,0)_*\mu_0$ and where μ_0 is the initial Wigner measure of coherent, Hermite and orthogonal states previously computed.

∎

Chapter 5

The general case with a two-body singular potential

5.1 Introduction

The mean field theory for many-body quantum systems is an extensively studied mathematical subject (see for instance [1, 18, 19, 29, 39, 42, 43, 47, 58, 68, 69] and [52, 63, 110] for more old results). The main addressed question in this field is the accuracy of the mean field approximation. While this problem is now well-understood for the most significant examples of quantum mechanics, it has no satisfactory general answer. The raison is that all the known results are concerned either with a specific model or a specific choice of quantum states. Our aim here is to show that the mean field approximation for bosonic systems is rather a general principle that depends very little on these above-mentioned specifications.

The Hamiltonian of many-boson systems have formally the following form

$$H_N = \sum_{i=1}^{N} A_i + \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} q_{i,j}^{(N)} = H_N^0 + q_N \,,$$

where A is a one particle kinetic energy and $q_{i,j}^{(N)}$ is a pair interaction potential between the i^{th} and j^{th} particles. It could be significant to include multi-particles interactions but to keep the presentation as simple as possible we don't do so (see [11, 78, 112]). Assume that H_N is a self-adjoint operator on some symmetric tensor product space $\bigvee^N \mathcal{Z}_0$. Then according to the Heisenberg equation the quantum dynamics yield the time-evolved states,

$$\varrho_N(t) := |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}| \,.$$

The mean-field approximation provides the first asymptotics of physical measurements in the state $\varrho_N(t)$ when the number of particles N is large. Precisely the approximation deals with the following quantities,

$$\lim_{N\to\infty} \mathrm{Tr}\left[\varrho_N(t)\, B \otimes 1^{\otimes(N-k)}\right],$$

where B is a given observable on the k first particles. Actually one can prove that, up to extracting a subsequence, there exists a Borel probability measure μ_0 on $\mathcal{Z}_0$ such that

$$\lim_{N\to\infty} \mathrm{Tr}\,[\varrho_N(0)\, B\otimes 1^{\otimes(N-k)}] = \int_{\mathcal{Z}_0} \langle z^{\otimes k}, Bz^{\otimes k}\rangle_{\vee^k \mathcal{Z}_0}\, d\mu_0(z)\,, \tag{5.1.1}$$

for any compact operator $B \in \mathcal{L}(\bigvee^k \mathcal{Z}_0)$, $1 \leqslant k \leqslant N$ (k is kept fixed while $N \to \infty$). Such a result is proved in [11] and in some sense it is related to a De Finetti quantum theorem [30, 77]. So this allows to understand the structure of the above limit (5.1.1) at time $t = 0$ and there is indeed no loss of generality if we suppose that (5.1.1) holds true for the sequence of states $(\varrho_N(0))_{N\in\mathbb{N}}$. Once this is observed then the mean-field approximation precisely says that for all times $t \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathrm{Tr}\,[\varrho_N(t)\, B\otimes 1^{\otimes(N-k)}] = \int_{\mathcal{Z}_0} \langle z^{\otimes k}, Bz^{\otimes k}\rangle_{\vee^k \mathcal{Z}_0}\, d\mu_t(z)\,, \tag{5.1.2}$$

where $\mu_t = \Phi(t,0)_*\mu_0$ is a push-forward measure, μ_0 is given by (5.1.1) and $\Phi(t,0)$ is the nonlinear flow which solves the mean-field classical equation on $\mathcal{Z}_0$,

$$i\partial_t z = Az + F(z)\,. \tag{5.1.3}$$

Here the nonlinear term $F(z)$ is related to the interaction q_N and the equation (5.1.3) provides the mean-field dynamics (for instance Hartree or NLS type equations). In this article we prove the statement (5.1.2) within an abstract framework and under general assumptions. It is common to express the mean-field limit with the language of reduced density matrices. So, we remark that (5.1.2) implies the convergence of reduced density matrices in the trace-class norm (see [11] for a proof of this fact). There are essentially two requirements for the accuracy of the mean field approximation. The first concerns the regularity of the states $\varrho_N(0)$ and the second deals with the criticality of the interaction q_N. So, we assume that the quantum states have asymptotically finite kinetic energy at time $t = 0$, i.e.:

$$\mathrm{Tr}\,[\varrho_N(0)\, A\otimes 1^{\otimes(N-1)}] \leqslant C\,, \tag{5.1.4}$$

uniformly in N (A is the one particle Kinetic energy). This is a reasonable requirement and in some sense a minimal one if we use energy type methods to deal with the quantum and classical dynamics. We give here only some informal insight on the assumption on q_N. Our main result is presented in detail in the next section and it is based on the abstract conditions (**D1**)-(**D2**). Suppose that A is the fractional Laplacian $(-\Delta)^s$, $s > 0$, in $L^2(\mathbb{R}^d)$ and the interaction q_N is given by

$$q_N := \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} W(x_i - x_j)\,, \quad x_i, x_j \in \mathbb{R}^d\,,$$

where $W : \mathbb{R}^d \to \mathbb{R}$ is a measurable function. Roughly speaking, our result says that the mean-field approximation holds true in general for states satisfying (5.1.4) if:

- The system is confined and the interaction W is subcritical.
- The interaction W is subcritical with some decay at infinity.
- The system is confined and the interaction W is critical.

If the system is not confined and the interaction W is critical then we do not expect the mean-field approximation to be true for any states with the regularity (5.1.4). However, if we are able to prove higher regularity on the quantum states $\varrho_N(t)$ for all times then it is possible to justify the mean-field limit as in the other cases. Here subcritical/critical means that the interaction W belongs to $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with subcritical/critical exponent with respect to the kinetic energy $A = (-\Delta)^s$ according to the Sobolev embedding $H^{\frac{s}{2}}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$. This emphasizes in particular the fact that the accuracy of the mean-field approximation depends very much on the criticality of the interaction and the regularity of initial states rather than the structure of the initial states or the exact model considered.

The method we use follows the one introduced in [12] which is based on general properties of Wick quantization in infinite dimensional spaces, Wigner measures and measure transportation techniques. We improve and simplify this method at several steps. For instance we consider only states $\varrho_N(0)$ in the symmetric tensor product $\bigvee^N \mathcal{Z}_0$ and avoid to work with states in the symmetric Fock space. This simplifies and strengthes the intermediate results. Moreover, a key argument related to convergence is clarified (see Section 5.5). The adaptation of measure transportation techniques in [5] to non-homogenous PDE was done in [12] with a somewhat strong condition on a related velocity field (see the assumption (**C2**) compared to the one used in [12]). This restricted the type of nonlinearity $F(z)$ that can be handled with this method. An improvement to a wider setting, briefly presented in Appendix B, is achieved in detail in [79].

As an illustration of the Wigenr measures techniques used in this article, we also recover a result proved in [77] concerning the limit of the ground state energy of H_N when the system of bosons is trapped.

Overview: In the following section our main result is presented in detail and illustrated with several examples. Self-adjointness and existence of the quantum dynamics is discussed in Section 5.3. The proof of our main Theorem 5.2.2 goes through three steps: A Duhamel's formula in Section 5.4, a convergence argument in Section 5.5 and a uniqueness result for a Liouville equation in Section 5.6. The technical tools used along the article are explained in Sections 2.1.4, 2.2.4 and 3.1 and concern respectively the Wick quantization, Wigner measures and transport along characteristic curves.

5.2 Preliminaries and results

In this section we introduce a general abstract setting suitable for the study of Hamiltonians of many-boson systems. Then we briefly recall the notion of Wigner measures and state the main results of the present article. We will often use conventional notations. In particular, the Banach space of bounded (resp. compact) operators from one Hilbert space $\mathfrak{h}_1$ into another one $\mathfrak{h}_2$ is denoted by $\mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_2)$ (resp. $\mathcal{L}^\infty(\mathfrak{h}_1, \mathfrak{h}_2)$). If C (resp. q) is an operator (resp. a quadratic form) on a Hilbert space then $D(C)$ (resp. $Q(q)$) denotes its domain. In particular, if C is a self-adjoint operator then $Q(C)$ denotes its form domain (i.e. the subspace $D(|C|^{\frac{1}{2}})$).

General framework: Let $\mathcal{Z}_0$ be a separable Hilbert space. The n-fold tensor product of $\mathcal{Z}_0$ is denoted by $\otimes^n \mathcal{Z}_0$. There is a canonical action $\sigma \in \Sigma_n \to \Pi_\sigma$ of the $n-$th symmetric group Σ_n on $\otimes^n \mathcal{Z}_0$ verifying

$$\Pi_\sigma f_1 \otimes ... \otimes f_n = f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_n} . \tag{5.2.1}$$

Hence each Π_σ extends to an unitary operator on $\otimes^n \mathcal{Z}_0$ with the relation $\Pi_\sigma \Pi_{\sigma'} = \Pi_{\sigma\circ\sigma'}$ satisfied for any $\sigma, \sigma' \in \Sigma_n$. Furthermore, the average of all these operators $(\Pi_\sigma)_{\sigma \in \Sigma_n}$, i.e.

$$\mathcal{S}_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \Pi_\sigma \,, \tag{5.2.2}$$

defines an orthogonal projection on $\otimes^n \mathcal{Z}_0$. By definition the symmetric n-fold tensor product of $\mathcal{Z}_0$ is the Hilbert subspace

$$\bigvee^n \mathcal{Z}_0 = \mathcal{S}_n(\mathcal{Z}_0^{\otimes n}) \,.$$

Consider now an operator A on $\mathcal{Z}_0$ and assume that:
Assumption (A1):

$$\textit{A is a non-negative and self-adjoint operator on $\mathcal{Z}_0$.} \tag{\textbf{A1}}$$

For $i = 1, \cdots, n$, let

$$A_i = 1^{\otimes(i-1)} \otimes A \otimes 1^{\otimes(n-i)},$$

where the operator A in the right hand side acts on the i^{th} component. The free Hamiltonian of a many-boson system is

$$H_N^0 = \sum_{i=1}^N A_i \,, \tag{5.2.3}$$

which is a self-adjoint non-negative operator on $\bigvee^N \mathcal{Z}_0$. In order to introduce a two particles interaction in an abstract setting we consider a symmetric quadratic form q on $Q(A_1 + A_2) \subset \otimes^2 \mathcal{Z}_0$. Here $A_1 + A_2$ is considered as an operator on $\otimes^2 \mathcal{Z}_0$ and the subspace $Q(A_1 + A_2)$ contains non-symmetric vectors. Throughout this paper we assume:
Assumption (A2):

$$\begin{array}{l} \textit{q is a symmetric sesquilinear form on $Q(A_1 + A_2)$ satisfying :} \\ \exists\, 0 < a < 1,\ b > 0, \quad \forall u \in Q(A_1 + A_2),\ |q(u,u)| \leqslant a\langle u, (A_1 + A_2)u\rangle + b\|u\|^2_{\otimes^2 \mathcal{Z}_0} \,. \end{array} \tag{\textbf{A2}}$$

As a consequence of the above assumption, q can be identified with a bounded operator $\tilde{q}$ satisfying the relation:

$$q(u,v) = \langle u, \tilde{q}\, v\rangle_{\otimes^2 \mathcal{Z}_0} \,, \quad \forall u, v \in Q(A_1 + A_2) \,, \tag{5.2.4}$$

and $\tilde{q}$ acts from the Hilbert space $Q(A_1 + A_2)$ equipped with the graph norm into its dual $Q'(A_1 + A_2)$ with respect to the inner product of $\otimes^2 \mathcal{Z}_0$.
Now, we define a collection of quadratic forms $(q_{i,j}^{(n)})_{1 \leqslant i < j \leqslant n}$ by

$$q_{i,j}^{(n)}(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n, \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) = q(\varphi_i \otimes \varphi_j, \psi_i \otimes \psi_j) \prod_{k \neq i,j} \langle \varphi_k, \psi_k \rangle \,, \tag{5.2.5}$$

for any $\varphi_1, \cdots, \varphi_n, \psi_1, \cdots, \psi_n$ in $Q(A)$. By linearity all the $q_{i,j}^{(n)}$ extend to well defined quadratic forms on the algebraic tensor product $\otimes^{alg,n} Q(A)$. Using the assumptions (**A1**)-(**A2**), we prove in Lemma 5.3.1 that each $q_{i,j}^{(n)}$, $1 \leqslant i < j \leqslant n$, extends uniquely to a symmetric quadratic form on

$Q(H_n^0) \subset \bigvee^n \mathcal{Z}_0$.
We now consider the *many-boson Hamiltonian* to be the quadratic form on $Q(H_N^0)$ given by

$$H_N = \sum_{i=1}^{N} A_i + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} q_{i,j}^{(N)} = H_N^0 + q_N . \tag{5.2.6}$$

Actually the assumptions (**A1**)-(**A2**) imply the existence of the many-boson dynamics, since there exists a unique self-adjoint operator, denote again by H_N, associated to the quadratic form (5.2.6) (see Proposition 5.3.4).

The classical dynamics: Let $(Q(A), \| \cdot \|_{Q(A)})$ be the domain form of the non-negative self-adjoint operator A equipped with the graph norm,

$$\|u\|_{Q(A)}^2 = \langle u, (A+1)u \rangle, \, u \in Q(A) \,,$$

and $Q'(A)$ its dual with respect to the inner product of $\mathcal{Z}_0$. The quadratic form q defines a quartic monomial,

$$z \in Q(A) \mapsto q_0(z) := \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2}) \,,$$

which is Gâteaux differentiable on $Q(A)$. Hence one can define the Gâteaux derivative of q_0 with respect to $\bar{z}$ according to the formula:

$$\partial_{\bar{z}} q_0(z)[u] = \frac{1}{2} \partial_{\bar{\lambda}} q((z+\lambda u)^{\otimes 2}, (z+\lambda u)^{\otimes 2})_{|\bar{\lambda}=0},$$

where $\partial_{\bar{\lambda}}$ is the Wirtinger derivative in the complex field $\mathbb{C}$. For each $z \in Q(A)$, the map $u \mapsto \partial_{\bar{z}} q_0(z)[u]$ is a anti-linear continuous form on $Q(A)$ and hence $\partial_{\bar{z}} q(z)$ can be identified with a vector $\partial_{\bar{z}} q_0(z) \in Q'(A)$ by the Riesz representation theorem. We shall assume the following assumptions on the classical mean field equation.
Assumption (C1): *The classical field equation*

$$\begin{cases} i\partial_t z = Az + \partial_{\bar{z}} q_0(z) \\ z_{|t=0} = z_0 \in Q(A), \end{cases} \tag{C1}$$

is globally well-posed on the form domain $Q(A)$. Moreover, the energy and the charge conservation hold true for initial data in $Q(A)$.
Global well-posedness here means existence and uniqueness of a global strong solution $t \mapsto z(t) \in \mathcal{C}^0(\mathbb{R}, Q(A)) \cap \mathcal{C}^1(\mathbb{R}, Q'(A))$ for each $z_0 \in Q(A)$ and continuous dependence on initial data (i.e.: The map $z_0 \mapsto z(\cdot) \in \mathcal{C}^0(I, Q(A))$ is continuous for any compact interval I containing 0). We have two formally conserved quantities, namely the charge $||z||_{\mathcal{Z}_0}$ and the classical energy:

$$h(z) = \langle z, Az \rangle + q_0(z) = \langle z, Az \rangle + \frac{1}{2} q(z \otimes z, z \otimes z) \,. \tag{5.2.7}$$

Remark that the main examples of the above field equation are the nonlinear Schrödinger and Hartree equations. We need also a second assumption.
Assumption (C2): *The vector field $\partial_{\bar{z}} q_0 \ : \ Q(A) \to Q'(A)$ satisfies:*

$$\exists C > 0, \, \forall z \in Q(A), \quad \|\partial_{\bar{z}} q_0(z)\|_{\mathcal{Z}_0} \leqslant C \|z\|_{Q(A)}^2 \|z\|_{\mathcal{Z}_0} \,. \tag{C2}$$

The Wigner measures: The mean-field problem is tackled here through the Wigner measures method elaborated in [9, 12]. The idea of these measures has its roots in the finite dimensional semi-classical analysis. It allows to generalize the notion of mean-field convergence to states that are not coherent nor factorized. For ease of reading, we briefly recall their definition here while their main features are discussed in Section 2.1.4.

Definition 5.2.1. *Let $\{\varrho_N := |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\vee^N \mathcal{Z}_0$, i.e $\|\Psi^{(N)}\|_{\vee^N \mathcal{Z}_0} = 1$. The set $\mathcal{M}(\varrho_N, N \in \mathbb{N})$ of Wigner measures of $(\varrho_N)_{N\in\mathbb{N}}$ is the set of Borel probability measures on $\mathcal{Z}_0$, μ, such that there exists a subsequence $(N_k)_{k\in\mathbb{N}}$ satisfying:*

$$\forall \xi \in \mathcal{Z}_0\,, \quad \lim_{k\to+\infty} \langle \Psi^{(N_k)}, \mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N_k)}\rangle = \int_{\mathcal{Z}_0} e^{2i\pi \operatorname{Re}\langle \xi, z\rangle}\, d\mu(z)\,, \tag{5.2.8}$$

where $\mathcal{W}(\sqrt{2}\pi\xi)$ is the Weyl operator in the symmetric Fock space defined in (**Weyl operators**) *with $\varepsilon = \frac{1}{N_k}$.*

The right hand side of (5.2.8) is the inverse Fourier transform of the measure μ. So Wigner measures are identified through their characteristic functions. Moreover, it was proved in [9, Theorem 6.2] that the set $\mathcal{M}(\varrho_N, N \in \mathbb{N})$ is nonempty and according to [10, 11, 12] it is a convenient tool for the study the mean-field limit. In particular, it allows to understood the convergence of reduced density matrices (5.1.2), which are the main analyzed quantities in other approaches ([110]).

5.2.1 Results

Dynamical result: Our first result concerns the effectiveness of the mean field approximation for general N-particle states and under general assumptions (**D1**)-(**D2**). We prove that the time-dependant Wigner measures of evolved states $\varrho_N(t) := |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}|$ are the push-forward of the initial measures (associated with the initial states $\varrho_N(0)$) by the global flow of the field equation in (**C1**). Eventually, if $\varrho_N(0)$ has only one Wigner measure then $\varrho_N(t)$ will have also one single Wigner measure described as above. The result is applicable to either trapped or untrapped systems of bosons.
Assumption (**D1**): *A has a compact resolvent and there exists a subspace D dense in $Q(A)$ such that for any $\xi \in D$,*

$$\begin{aligned} &\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + \lambda)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0, \mathcal{Z}_0)} = 0\,, \\ &\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+\lambda)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + 1)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0, \mathcal{Z}_0)} = 0\,. \end{aligned} \tag{\textbf{D1}}$$

Our second main assumption is given below and it implies the two limits in (**D1**).
Assumption (**D2**): *There exists a subspace D dense in $Q(A)$ such that for any $\xi \in D$,*

$$\langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^\infty(\bigvee^2 \mathcal{Z}_0, \mathcal{Z}_0)\,. \tag{\textbf{D2}}$$

Consider the abstract setting explained above with a separable Hilbert space $\mathcal{Z}_0$, a one-particle self-adjoint operator A and a two-body interaction q. Then our main result on the dynamical mean-field problem is stated below.

Theorem 5.2.2. *Assume* (**A1**)-(**A2**) *and* (**C1**)-(**C2**) *and suppose that either* (**D1**) *or* (**D2**) *holds true. Let* $\{\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ *a sequence of normal states on* $\bigvee^N \mathcal{Z}_0$ *with a unique Wigner measure* μ_0 *and satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N}, \quad \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle \leqslant CN\,. \tag{5.2.9}$$

Then for any time $t \in \mathbb{R}$ *the family* $\{\varrho_N(t) = |e^{-itH_N}\Psi^{(N)}\rangle\langle e^{-itH_N}\Psi^{(N)}|\}_{N\in\mathbb{N}}$ *has a unique Wigner measure* μ_t *which is a Borel probability measure on* $Q(A)$*. In addition* $\mu_t = \Phi(t,0)_*\mu_0$*, the push-forward of the initial measure* μ_0 *by the globally well-defined flow* $\Phi(t,0)$ *on* $Q(A)$ *associated to the field equation:*

$$\begin{cases} i\partial_t z = Az + \partial_{\bar z} q_0(z) \\ z_{|t=0} = z_0. \end{cases} \tag{5.2.10}$$

Remarks 5.2.3. *1) The above theorem remains true if we assume that* A *is semi-bounded from below.*
2) It is not necessary to assume that ϱ_N *admits a unique Wigner measure* μ_0*. In general the result says:*

$$\mathcal{M}(\varrho_N(t), N \in \mathbb{N}) = \{\Phi(t,0)_*\mu_0, \mu_0 \in \mathcal{M}(\varrho_N, N\in\mathbb{N})\}\,.$$

3) Without essential changes in the proof of Theorem 5.2.2, we can suppose that ϱ_N *is an arbitrary sequence of normal states in* $\bigvee^N \mathcal{Z}_0$ *satisfying:*

$$\exists C > 0, \forall N \in \mathbb{N}, \quad \mathrm{Tr}\,[\varrho_N H_N^0] \leqslant CN\,.$$

Variational result: Our second result concerns the ground state energy of trapped many-boson systems in the mean-field limit. Consider the Hamiltonian H_N given by (5.2.6) and suppose that (**A1**)-(**A2**) are satisfied. The confinement of the system is equivalent to the requirement that the operator A has a compact resolvent. By definition the quantum ground state energy is

$$E(N) := \inf_{\substack{\Psi^{(N)}\in Q(H_N^0) \\ \|\Psi^{(N)}\|_{\bigvee^N \mathcal{Z}_0}=1}} \langle \Psi^{(N)}, H_N \Psi^{(N)}\rangle\,.$$

On the other hand the classical energy functional is

$$h(z) = \langle z, Az\rangle + \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2})\,, \qquad \forall z \in Q(A)\,.$$

Using (**A2**), one observes that $\inf_{z\in Q(A), \|z\|_{\mathcal{Z}_0}=1} h(z)$ is finite. In fact for any $z \in Q(A)$ such that $\|z\|_{\mathcal{Z}_0} = 1$,

$$h(z) \geqslant (1-a)\langle z^{\otimes 2}, (A_1 + A_2) z^{\otimes 2}\rangle - C_1 \geqslant -C_1.$$

Theorem 5.2.4. *Assume* (**A1**)-(**A2**) *and suppose that* A *has a compact resolvent. Then*

$$\lim_{N\to+\infty} \frac{E(N)}{N} = \lim_{N\to+\infty} \frac{1}{N} \inf_{\substack{\Psi^{(N)}\in Q(H_N^0) \\ \|\Psi^{(N)}\|=1}} \langle \Psi^{(N)}, H_N \Psi^{(N)}\rangle = \inf_{\substack{z\in Q(A) \\ \|z\|_{\mathcal{Z}_0}=1}} h(z) > -\infty.$$

5.2.2 Examples

In this section we provide several examples to which the general result of Theorem 5.2.2 is applicable. But first we observe that the two limits in (**D1**) are satisfied whenever q is infinitesimally $A_1 + A_2$-form bounded. This indeed allows to handle the situation when the interaction is subcritical. But when the interaction is comparable to the kinetic energy we rely directly on (**D1**) which seems to be the appropriate assumption in this case.

Lemma 5.2.5. *Assume* (**A1**)-(**A2**) *and suppose that the quadratic form q is infinitesimally A_1+A_2-form bounded. Then for any $\xi \in Q(A)$,*

$$\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + \lambda)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0,\mathcal{Z}_0)} = 0\,,$$
$$\lim_{\lambda\to+\infty} ||\langle\xi| \otimes (A+\lambda)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + 1)^{-\frac{1}{2}}||_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0,\mathcal{Z}_0)} = 0\,.$$

Proof. Let $\Phi \in \mathcal{Z}_0$ and $\Psi \in \bigvee^2 \mathcal{Z}_0$ then by Cauchy-Schwarz inequality,

$$\begin{aligned}|\langle\Phi, \langle\xi| \otimes (A+1)^{-\frac{1}{2}}\, \mathcal{S}_2\, \tilde{q}(A_1 + A_2 + \lambda)^{-\frac{1}{2}}\, \Psi\rangle| &= |q(\mathcal{S}_2\xi \otimes (A+1)^{-\frac{1}{2}}\Phi, (A_1 + A_2 + \lambda)^{-\frac{1}{2}}\,\Psi)| \\ &\leqslant |q(\mathcal{S}_2\xi \otimes (A+1)^{-\frac{1}{2}}\Phi)|^{\frac{1}{2}}\ \ |q((A_1 + A_2 + \lambda)^{-\frac{1}{2}}\,\Psi)|^{\frac{1}{2}}\,,\end{aligned}$$

with $q(u) = q(u, u)$. Remark that $|q(\mathcal{S}_2\xi \otimes (A+1)^{-\frac{1}{2}}\Phi)|$ is bounded thanks to (**A2**) and the fact that $\xi \in Q(A)$. Since q is infinitesimally $A_1 + A_2$-form bounded, then for any $\alpha > 0$ there exists $C(\alpha) > 0$ such that

$$\begin{aligned}|q((A_1 + A_2 + \lambda)^{-\frac{1}{2}}\,\Psi)| &\leqslant \alpha\langle\Psi, (A_1 + A_2 + \lambda)^{-1}(A_1 + A_2 + \frac{C(\alpha)}{\alpha})\,\Psi\rangle \\ &\leqslant \max(\alpha, \frac{C(\alpha)}{\lambda})\,\|\Psi\|\,.\end{aligned}$$

This proves the first limit in (**D1**) when $\lambda \to \infty$. The second one follows by a similar argument. ∎

All the examples listed below are covered by Theorem 5.2.2.

Example 5.2.6 (The two-body delta interaction)**.** *Non-relativistic systems of trapped bosons with a two-body point interaction,*

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{\kappa}{N} \sum_{1\leqslant i<j\leqslant N} \delta(x_i - x_j), \qquad x_i, x_j \in \mathbb{R},\ \kappa \in \mathbb{R}, \tag{5.2.11}$$

where δ is the Dirac distribution and V is a real-valued potential which splits into two parts $V = V_1+V_2$ such that

$$V_1 \in L^1_{loc}(\mathbb{R}),\ V_1 \geqslant 0\,,\ \lim_{|x|\to+\infty} V_1(x) = +\infty\,,$$
V_2 *is* $-\Delta$*-form bounded with a relative bound less than one.*

This model has been studied for instance in [1, 6]. The operator $A = -\Delta + V$ *is self-adjoint semi-bounded from below and* A *has a compact resolvent according to [100, Theorem X19]. The two-body interaction* q *is given by* $q(z^{\otimes 2}, z^{\otimes 2}) = \kappa\langle z^{\otimes 2}, \delta(x_1 - x_2) z^{\otimes 2}\rangle = \kappa \|z\|^4_{L^4(\mathbb{R})}$ *and satisfies for any* $u \in Q(A_1 + A_2)$,

$$\forall \alpha > 0,\ |q(u,u)| \leqslant \frac{\alpha\kappa}{2\sqrt{2}}\langle u, A_1 + A_2\, u\rangle + \frac{\kappa}{4\alpha\sqrt{2}}\|u\|^2_{L^2(\mathbb{R}^2)}.$$

For a detailed proof of the latter inequality see [6, Lemma A.1]. Hence (**A1**)-(**A2**) *are verified and by Lemma 5.2.5 the assumption* (**D1**) *holds true. The vector field is given by* $\partial_{\bar z} q_0(z) = \kappa|z|^2 z :\ Q(A) \to Q(A)$ *and satisfies the inequalities,*

$$\forall x, y \in Q(A),\ \exists C := C(\|x\|_{Q(A)}, \|y\|_{Q(A)}) > 0;\ \||x|^2 x - |y|^2 y\|_{Q(A)} \leqslant C\|x - y\|_{Q(A)}\,,$$

and

$$\forall z \in Q(A),\ \||z|^2 z\|_{L^2(\mathbb{R})} \leqslant C\|z\|^2_{H^1(\mathbb{R})}\|z\|_{L^2(\mathbb{R})} \leqslant C\|z\|^2_{Q(A)}\|z\|_{L^2(\mathbb{R})}\,, \tag{5.2.12}$$

since the inclusion $Q(A) \subset H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ *holds by Sobolev embedding and the fact that* $Q(A) = \{u \in L^2(\mathbb{R}), u' \in L^2(\mathbb{R}), V_1^{\frac{1}{2}} u \in L^2(\mathbb{R})\}$. *Therefore the vector field* $\partial_{\bar z} q_0(z)$ *is locally Lipschitz on* $Q(A)$ *and the NLS equation*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + \kappa|z|^2 z \\ z_{|t=0} = z_0, \end{cases} \tag{\textbf{NLS}}$$

is locally well-posed on $Q(A)$. *Furthermore, using the energy and charge conservation one shows the global well-posedness of the* **NLS** *equation.*

Example 5.2.7 (Trapped bosons)**.** *Non relativistic trapped many-boson systems with singular two-body potential:*

$$H_N = \sum_{i=1}^N -\Delta_{x_i} + V(x_i) + \frac{1}{N}\sum_{1\leqslant i<j\leqslant N} W(x_i - x_j), \quad x_i, x_j \in \mathbb{R}^d. \tag{5.2.13}$$

where V *is a real-valued potential which splits into two parts* $V = V_1 + V_2$ *such that:*

$$\begin{aligned} &V_1 \in C^\infty(\mathbb{R}^d, \mathbb{R}),\ V_1 \geqslant 0,\ D^\alpha V_1 \in L^\infty(\mathbb{R}^d),\ \forall \alpha \in \mathbb{N}^d, |\alpha| \geqslant 2\,,\\ &V_1(x) \to \infty,\ \text{ when } |x| \to \infty\,,\\ &V_2 \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ p \geqslant 1,\ p > \frac{d}{2}\,, \end{aligned}$$

and $W :\ \mathbb{R}^d \to \mathbb{R}$ *is an even measurable function verifying:*

$$W \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ q \geqslant 1,\ q \geqslant \frac{d}{2}, (\text{and } q > 1 \text{ if } d = 2). \tag{5.2.14}$$

By Gagliardo-Nirenberg inequality we know that (5.2.14) *implies that* W *is infinitesimally* $-\Delta$*-form bounded. So, the assumptions* (**A1**)-(**A2**) *and* (**D1**) *are satisfied. Moreover, the vector field* $[\partial_{\bar z} q_0](z) = W * |z|^2 z :\ Q(A) \to Q'(A)$ *satisfies for any* $z \in Q(A)$,

$$\|W * |z|^2 z\|_{L^2(\mathbb{R}^d)} \leqslant \|(-\Delta + 1)^{-\frac{1}{2}} W (-\Delta + 1)^{-\frac{1}{2}}\|\ \|z\|^2_{H^1(\mathbb{R}^d)}\ \|z\|_{L^2(\mathbb{R}^d)}\,. \tag{5.2.15}$$

The global well-posedness on $Q(A)$, conservation of energy and charge of the Hartree equation

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{t=0} = z_0, \end{cases} \tag{Hartree}$$

are proved in [28] Theorem 9.2.6 and Remark 9.2.8. Observe that the assumption on W are satisfied by the Coulomb type potentials $\frac{\lambda}{|x|^\alpha}$ when $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$.

Example 5.2.8 (Untrapped bosons)**.** *Non-relativistic untrapped many-boson systems,*

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V(x_i) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} W(x_i - x_j), \quad x_i, x_j \in \mathbb{R}^d.$$

where the potentials V and W satisfy the following assumptions for some p and q,

$$\begin{aligned} &V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ p \geqslant 1,\ \ p > \frac{d}{2}, \\ &W \in L^q(\mathbb{R}^d) + L^\infty_0(\mathbb{R}^d),\ q \geqslant 1,\ \ q \geqslant \frac{d}{2}, (\textit{and } q > 1 \textit{ if } d = 2) \end{aligned} \tag{5.2.16}$$

Here $L^\infty_0(\mathbb{R}^d)$ denotes the space of bounded measurable functions going to 0 at infinity. For instance Coulomb potentials $\frac{\lambda}{|x|^\alpha}$ for $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$ satisfy (5.2.16)*. As in the previous example* (**A1**)*-*(**A2**) *are satisfied and* (**D2**) *is verified if we check that $(1-\Delta_x)^{-\frac{1}{2}} W(x) (1-\Delta_x)^{-\frac{1}{2}}$ is compact (see the proof of [12, Lemma 3.10]). In fact W decomposes as $W = W_1 + W_2$ with $W_1 \in L^q(\mathbb{R}^d)$ and $W_2 \in L^\infty_0(\mathbb{R}^d)$. We know that $W_2(1-\Delta)^{-\frac{1}{2}} \in \mathcal{L}^\infty(L^2(\mathbb{R}^d))$ (see for instance [66, Proposition 3.21]). Therefore we only need to check that $(1-\Delta_x)^{-\frac{1}{2}} W_1(x)(1-\Delta_x)^{-\frac{1}{2}}$ is compact. Let $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ such that $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ in a neighborhood of 0. We denote $\chi_m(x) := \chi(\frac{x}{m})$, for $x \in \mathbb{R}^d$ and $m \in \mathbb{N}^*$. For a given measurable function g let $(g^\delta)_{\delta > 0}$ denotes*

$$g^\delta = \begin{cases} g, \ \textit{if } |g| < \delta \\ \delta, \ \textit{if } g \geqslant \delta \\ -\delta, \ \textit{if } g \leqslant -\delta\,. \end{cases} \tag{5.2.17}$$

Writing the decomposition

$$W_1 = \underbrace{(\chi_m W_1)^\delta}_{L^\infty_0(\mathbb{R}^d)} + \underbrace{W_1 - (\chi_m W_1)^\delta}_{L^q(\mathbb{R}^d)},$$

we observe that

$$(1-\Delta)^{-\frac{1}{2}} (\chi_m W_1)^\delta (1-\Delta)^{-\frac{1}{2}} \in \mathcal{L}^\infty(L^2(\mathbb{R}^d)),$$

and for $\delta \to +\infty$ and $m \to +\infty$,

$$(1-\Delta)^{-\frac{1}{2}} (\chi_m W_1)^\delta (1-\Delta)^{-\frac{1}{2}} \longrightarrow (1-\Delta)^{-\frac{1}{2}} W_1 (1-\Delta)^{-\frac{1}{2}}, \tag{5.2.18}$$

in the norm topology. Hence (**D2**) *holds true. The convergence* (5.2.18) *is justified by the Gagliardo-Nirenberg's inequality,*

$$|\langle u, \big[(\chi_m W_1)^\delta - W_1\big] u\rangle| \leqslant C \|(\chi_m W_1)^\delta - W_1\|_{L^q(\mathbb{R}^d)} \, \|\nabla u\|^{2\alpha}_{L^2(\mathbb{R}^d)} \, \|u\|^{2(1-\alpha)}_{L^2(\mathbb{R}^d)}, \qquad \alpha = \frac{d}{2q}.$$

As in Example 5.2.7 the vector field $\partial_{\bar{z}} q_0(z)$ satisfies the inequality (5.2.15) *and the global well-posedness on $H^1(\mathbb{R}^d)$, conservation of energy and charge of the Hartree equation,*

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{|t=0} = z_0, \end{cases}$$

holds according to [28] Corollary 4.3.3 and Corollary 6.1.2.

Example 5.2.9 (Non-relativistic Bosons with magnetic field)**.** *Non-relativistic many-boson systems with an external magnetic field $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$ and an external electric field $V : \mathbb{R}^d \to \mathbb{R}$ are described by the Hamiltonian,*

$$H_N = \sum_{j=1}^{N} \left[(-i\nabla_{x_j} + \mathcal{A}(x_j))^2 + V(x_j)\right] + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} W(x_i - x_j), \tag{5.2.19}$$

where $W(x)$ is an even measurable function satisfying with $\mathcal{A}$ and V the assumptions:

$$\begin{aligned} & d \geqslant 3, \\ & \mathcal{A} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d), \\ & V \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}),\ V_+(x) \to \infty,\ \textit{when } |x| \to \infty\,, \\ & V_- \textit{ is } -\Delta\textit{-form bounded with relative bound less than } 1, \\ & W \in L^q(\mathbb{R}^d, \mathbb{R}) + L^\infty(\mathbb{R}^d, \mathbb{R}),\ \nabla W \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \textit{ for some } q > \frac{d}{2},\, p \geqslant \frac{d}{3}. \end{aligned}$$

Here $V_\pm$ denotes the positive and negative part of the potential V. Let $\nabla_{\mathcal{A}} := \nabla + i\mathcal{A}$ then the quadratic form

$$H_V(\mathcal{A})[f, g] := \int_{\mathbb{R}^d} \overline{\nabla_{\mathcal{A}} f(x)}\, \nabla_{\mathcal{A}} g(x)\, dx + \int_{\mathbb{R}^d} V(x) \overline{f(x)} g(x)\, dx\,,$$

defined on the form domain

$$\mathcal{H}^1_{\mathcal{A},V}(\mathbb{R}^d) := \{\varphi \in L^2(\mathbb{R}^d),\ \nabla_{\mathcal{A}}\, \varphi,\ V_+^{\frac{1}{2}} \varphi \in L^2(\mathbb{R}^d)\},$$

is closed and bounded from below and hence it defines a unique semi-bounded from below self-adjoint operator denoted $H_V(\mathcal{A})$ (see [14],[72]). Moreover, $C_0^\infty(\mathbb{R}^d)$ is a form core for $H_V(\mathcal{A})$. Hence (**A1**) *is true and since W satisfies the condition* (5.2.14) *of Example 5.2.7 we know that $W(x_1 - x_2)$ is infinitesimally $-\Delta_{x_1} - \Delta_{x_2}$-form bounded. Applying [14, Theorem 2.5] one concludes that $W(x_1 - x_2)$ is infinitesimally $H_0(\mathcal{A}) \otimes 1 + 1 \otimes H_0(\mathcal{A})$-form bounded and subsequently it is infinitesimally $H_V(\mathcal{A}) \otimes 1 + 1 \otimes H_V(\mathcal{A})$-form bounded. Hence* (**A2**) *is also true. Moreover, according to [14, Theorem 2.7] $H_V(\mathcal{A})$ has a compact resolvent and so assumption* (**D1**) *is satisfied.*
The global well-posedness in $\mathcal{H}^1_{\mathcal{A},V}(\mathbb{R}^d)$ of the Hartree equation with magnetic field

$$\begin{cases} i\partial_t z = (-i\nabla + \mathcal{A})^2 z + Vz + W * |z|^2 z \\ z_{|t=0} = z_0, \end{cases} \tag{5.2.20}$$

is proved in [91] together with energy and charge conservation. Furthermore, the assumption (**C2**) *holds since*

$$\|\big[W_1 * \bar{z}z\big]z\|_{L^2(\mathbb{R}^d)} \leqslant \|W_1\|_{L^q(\mathbb{R}^d)}\|\,|z|^2\|_{L^{\frac{q}{q-1}}(\mathbb{R}^d)}\|z\|_{L^2(\mathbb{R}^d)} \leqslant \|W_1\|_{L^q(\mathbb{R}^d)}\|z\|^2_{H^1_{\mathcal{A},V}(\mathbb{R}^d)}\|z\|_{L^2(\mathbb{R}^d)}. \quad (5.2.21)$$

Here $W = W_1 + W_2$ with $W_1 \in L^q(\mathbb{R}^d)$ and $W_2 \in L^\infty(\mathbb{R}^d)$. The mean field problem for this type of model was studied in [88].

Example 5.2.10 (Semi-relativistic bosons with critical interaction)**.** *Semi-relativistic systems of bosons have the many-body Hamiltonian*

$$H_N = \sum_{j=1}^N \sqrt{-\Delta_{x_j} + m^2} + V(x_j) + \frac{\lambda}{N}\sum_{1\leqslant i<j\leqslant N}\frac{1}{|x_i - x_j|}, \quad x_i, x_j \in \mathbb{R}^3\,,$$

with $0 \leqslant \lambda < \lambda_{cr}$, $\lambda_{cr}^{-1} := 2\lim_{\alpha\to\infty}||\frac{1}{|x|}(-\Delta+\alpha)^{-\frac{1}{2}}||$, $m \geqslant 0$ and V is real-valued measurable function $V = V_1 + V_2$ satisfying

$$V_1 \in L^1_{loc}(\mathbb{R}^3),\ V_1 \geqslant 0,\ V_1(x) \to \infty \text{ when } |x| \to \infty\,,$$
$$V_2 \text{ is } \sqrt{-\Delta} - \text{form bounded with a relative bound less than } 1\,.$$

The quadratic form

$$A[u,u] = \langle u, \sqrt{-\Delta + m^2}\, u\rangle + \langle u, Vu\rangle\,,$$
$$Q(A) = \{u \in L^2(\mathbb{R}^3),\ (-\Delta+m^2)^{\frac{1}{4}}u \in L^2(\mathbb{R}^3),\ V_1^{\frac{1}{2}}u \in L^2(\mathbb{R}^3)\}\,,$$

is semi-bounded from below and closed. So it defines a unique self-adjoint operator denoted by A. In particular assumption (**A1**) *is verified and* (**A2**) *is satisfied thanks to a Hardy type inequality (see for instance [12, Proposition D.3]). Hence the critical value λ_{cr} is finite and we have the inequality,*

$$||\frac{1}{|x|} * |z|^2\, z||_{L^2(\mathbb{R}^3)} \leqslant C||z||^2_{H^{1/2}(\mathbb{R}^3)}\,||z||_{L^2(\mathbb{R}^3)}\,.$$

Furthermore, Rellich's criterion shows that A has a compact resolvent. To prove the two limits in (**D1**)*, we use the following argument. For any $\xi, \Phi \in C_0^\infty(\mathbb{R}^3)$ and $\Psi \in C_0^\infty(\mathbb{R}^3)$,*

$$|\langle\Phi, \langle\xi| \otimes 1\mathcal{S}_2\frac{1}{|x-y|}\Psi\rangle| \leqslant ||\Phi||_{L^3(\mathbb{R}^3)}\,||T\Psi||_{L^{3/2}(\mathbb{R}^3)} \leqslant ||\Phi||_{H^{1/2}(\mathbb{R}^3)}\,||T\Psi||_{L^{3/2}(\mathbb{R}^3)}\,, \quad (5.2.22)$$

where T is the operator given by

$$T\Psi(y) := \int_{\mathbb{R}^3}\bar{\xi}(x)\,\frac{1}{|x-y|}\,\Psi(x,y)\,dx\,.$$

Using Hölder's inequality twice with the pairs (p,q), $2 < q < 3$, $\frac{3}{2} < p < 2$ and $(4, \frac{4}{3})$,

$$\begin{aligned} ||T\Psi(y)||^{3/2}_{L^{3/2}(\mathbb{R}^3)} &\leqslant \int_{\mathbb{R}^3}\left||\xi|^p * \frac{1}{|\cdot|^p}\right|^{\frac{3}{2p}} \times \left(\int_{\mathbb{R}^3}|\Psi(x,y)|^q\,dx\right)^{\frac{3}{2q}}\,dy \\ &\leqslant \left\||\xi|^p * \frac{1}{|\cdot|^p}\right\|^{3/2p}_{L^{6/p}(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3}||\Psi(\cdot,y)||^2_{L^q(\mathbb{R}^3)}\,dy\right)^{\frac{3}{4}}. \end{aligned}$$

By the fractional Gagliardo-Nirenberg's inequality in [61, Corollary 2.4], we see for $0 < \alpha < 1$ *and* $q = \frac{6}{3-\alpha}$,

$$||T\Psi(y)||^{3/2}_{L^{3/2}(\mathbb{R}^3)} \leqslant \left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|^{3/2p}_{L^{6/p}(\mathbb{R}^3)} ||\Psi(.,y)||^{2(1-\alpha)}_{L^2(\mathbb{R}^3)} \; ||(-\Delta)^{\frac{1}{4}}\Psi(.,y)||^{2\alpha}_{L^2(\mathbb{R}^3)} .$$

Therefore, using the inequality $a^\alpha b^{(1-\alpha)} \leqslant \varepsilon a + \varepsilon^{-\frac{\alpha}{1-\alpha}} b$ *for any* $\varepsilon, a, b > 0$, *we get*

$$||T\Psi(y)||^{3/2}_{L^{3/2}(\mathbb{R}^3)} \leqslant \left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|^{3/2p}_{L^{6/p}(\mathbb{R}^3)} \left(\varepsilon \langle \Psi, \sqrt{-\Delta_x}\Psi \rangle_{L^2(\mathbb{R}^6)} + \varepsilon^{-\frac{\alpha}{1-\alpha}} ||\Psi||^2_{L^2(\mathbb{R}^6)} \right) \tag{5.2.23}$$

Remark that Hardy-Littlewood-Sobolev's inequality yields

$$\left\| |\xi|^p * \frac{1}{|\cdot|^p} \right\|_{L^{6/p}(\mathbb{R}^3)} \leqslant C \, ||\xi||^p_{L^{\frac{6p}{6-p}}} < \infty. \tag{5.2.24}$$

So the inequalities (5.2.22),(5.2.23),(5.2.24), *provide*

$$|\langle \Phi, \langle \xi| \otimes (A+1)^{-\frac{1}{2}} \mathcal{S}_2 \frac{1}{|x-y|} (A+\lambda)^{-\frac{1}{2}} \otimes 1 \Psi \rangle| \leqslant C \big[\varepsilon + \frac{\varepsilon^{-\frac{\alpha}{1-\alpha}}}{\lambda} \big] \, ||\xi||^p_{L^{\frac{6p}{6-p}}} \; ||\Phi||_{L^2(\mathbb{R}^3)} \; ||\Psi||_{L^2(\mathbb{R}^6)} .$$

This proves the first limit when $\lambda \to \infty$, *the second one is similar and it is left for the reader.*
The global well-posedness in $Q(A)$, *conservation of energy and charge of the semi-relativistic Hartree equation*

$$\begin{cases} i\partial_t z = \sqrt{-\Delta + m^2} \; z + V(x) z + \dfrac{\lambda}{|x|} * |z|^2 z \\ z_{|t=0} = z_0 . \end{cases}$$

are proved in [73, Theorem 4] for all $\lambda \geqslant 0$. *The arguments used here extend also to non-relativistic systems of bosons with a critical interaction* $W(x-y) = \frac{\lambda}{|x-y|^2}$. *The condition* $\lambda < \lambda_{cr}$ *can also be removed with few modifications on the method employed in this article.*

Example 5.2.11 (The LLL-model)**.** *The LLL-model is related to the modeling of rapidly rotating Bose-Einstein condensates in the Lowest Landau Level (LLL). We briefly recall it here and refer the reader to [2] for more details. Here the scaling is different from the one developped in Subsection 2.1.1. The one particle state is the Bargmann space given by*

$$\mathscr{Z}_0 = \{ f \in L^2(\mathbb{C}_\xi, e^{-\frac{|\xi|^2}{h}} L(d\xi)), \; \partial_{\bar{\xi}} f = 0 \},$$

where $h > 0$ *is a small parameter and* $L(d\xi)$ *is the Lebesgue measure on* $\mathbb{C}$. *The space* $\mathscr{Z}_0$ *is equipped with the norm,*

$$||f||^2_{\mathscr{Z}_0} = \int_{\mathbb{C}} |f(\xi)|^2 \, e^{-\frac{|\xi|^2}{h}} \, \frac{L(d\xi)}{\pi h} \, ,$$

and it is a closed subspace of $L^2(\mathbb{C}, e^{-\frac{|\xi|^2}{h}} L(d\xi))$ *with a related orthogonal projection given explicitly by*

$$\begin{aligned} \Pi_h : L^2(\mathbb{C}, e^{-\frac{|\xi|^2}{h}} L(d\xi)) &\to \mathcal{Z}_0 \\ g \mapsto \Pi_h(g)(\xi) &= \int_{\mathbb{C}} e^{\frac{\xi.\bar{\tau}-|\tau|^2}{h}} \, g(\tau) \, \frac{L(d\tau)}{\pi h}. \end{aligned}$$

The many-body Hamiltonian describing the LLL-model in the mean field regime is

$$H_N = \sum_{i=1}^{N} 2(h\, \xi_i \partial_{\xi_i} + h) + \frac{1}{N} \sum_{1 \leqslant i,j \leqslant N} q_{i,j}^{(N)}$$

where q *is the quadratic form defined on* $\otimes^2 \mathcal{Z}_0$ *by*

$$q(u,u) = 2\alpha \int_{\mathbb{C}^2} \overline{u(\xi_1,\xi_2)}\, u(\frac{\xi_1+\xi_2}{2}, \frac{\xi_1+\xi_2}{2})\, e^{-\frac{|\xi_1|^2+|\xi_2|^2}{h}} \, \frac{L(d\xi_1 d\xi_2)}{(\pi h)^2} \, .$$

Using hypercontractivity inequalities (see [2, 92]), one can prove that q *is a bounded quadratic form on* $\bigvee^2 \mathcal{Z}_0$*. Hence the assumption* (**A2**) *is satisfied and since the operator* $\xi\, \partial_\xi$ *is unitary equivalent to the harmonic oscillator the assumptions* (**A1**) *and* (**D1**) *are also true.*
The classical field equation is determined by the following energy functional

$$h_{MF}(f) = \langle f, 2\big[h\xi\partial_\xi + h\big] f\rangle + \alpha \int_{\mathbb{C}} |f(\xi)|^4 e^{-\frac{2|\xi|^2}{h}} L(d\xi), \tag{5.2.25}$$

and the mean field dynamics associated with the LLL-model is given by

$$\begin{cases} i\partial_t f = 2(h\xi\partial_\xi + h) f + 2\alpha \Pi_h(e^{-\frac{|\xi|^2}{h}} |f|^2) f \\ f_{|t=0} = f_0. \end{cases} \tag{5.2.26}$$

Denote $\mathcal{F}_h^s$ *the space of entire functions such that*

$$\int_{\mathbb{C}} (1+|z|^2)^s |f(z)|^2 e^{-\frac{|z|^2}{h}} L(dz) < +\infty.$$

The global well-posedness of (5.2.26) *on the space* $\mathcal{F}_h^s$ *has been obtained in [92] for* $s = 2$ *together with energy and charge conservation (see [11] for* $s = 0$*). Actually, the quantity* $\langle f, h\, \xi\partial_\xi f\rangle$ *is also preserved for initial data in* $f_0 \in \mathcal{F}_h^2$ *hence one can show that* (5.2.26) *is well-posed in* $\mathcal{F}_h^1$ *with energy and charge conservation.*

5.3 Properties of the Quantum Dynamics

In this section we show that under the assumptions (**A1**)-(**A2**) the quadratic form (5.2.6) defines a unique self-adjoint operator H_N. Thereafter, a useful regularity property of the related quantum dynamics is stated in Proposition 5.3.5.

5.3.1 Selfadjointness

Remember that the quadratic form q satisfies (**A2**) and $q_{i,j}^{(n)}$, q_N are defined respectively by (5.2.5) and (5.2.6).

Lemma 5.3.1. *Assume* (**A1**)-(**A2**). *Then, for any* $1 \leqslant i < j \leqslant n$, $q_{i,j}^{(n)}$ *extends to a symmetric quadratic form on* $Q(A_i + A_j) \subset \otimes^n \mathcal{Z}_0$. *Moreover, for any* $\Phi \in Q(A_i + A_j)$,

$$|q_{i,j}^{(n)}(\Phi^{(n)}, \Phi^{(n)})| \quad \leqslant \quad a\langle \Phi^{(n)}, A_i + A_j\, \Phi^{(n)} \rangle + b\, \|\Phi^{(n)}\|^2_{\otimes^n \mathcal{Z}_0} \,. \tag{5.3.1}$$

Proof. Once the estimate (5.3.1) is proved for any $\Phi^{(n)} \in \otimes^{alg,n} Q(A)$, the extension of $q_{i,j}^{(n)}$ to the domain $Q(A_i + A_j)$ is straightforward since $\otimes^{alg,n} Q(A)$ is a form core for $A_i + A_j$. A simple computation yields for any $\Phi^{(n)}, \Psi^{(n)} \in \otimes^{alg,n} Q(A)$,

$$q_{i,j}^{(n)}(\Phi^{(n)}, \Psi^{(n)}) = q_{1,2}^{(n)}(\Pi_{(i,j)} \Phi^{(n)}, \Pi_{(i,j)} \Psi^{(n)})\,, \tag{5.3.2}$$

where $\Pi_{(i,j)}$ is the interchange operator defined in (5.2.1) with $\sigma = (i,j)$ is the particular permutation

$$(i,j) = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ i & j & \cdots & 1 & \cdots & 2 & \cdots & n \end{pmatrix}.$$

Moreover, one remarks that

$$\langle \Pi_{(i,j)} \Phi^{(n)}, A_1 + A_2\, \Pi_{(i,j)} \Psi^{(n)} \rangle = \langle \Phi^{(n)}, A_i + A_j\, \Psi^{(n)} \rangle .$$

Hence, it is enough to prove (5.3.1) for $i = 1$ and $j = 2$ and $\Phi^{(n)} \in \otimes^{alg,n} Q(A)$. Let $\{e_k\}_{k \in \mathbb{N}}$ be an O.N.B of $\mathcal{Z}_0$ such that $e_k \in Q(A)$ for all $k \in \mathbb{N}$. For $r \in \mathbb{N}^n$, $r = (r_1, \cdots, r_n)$, we denote

$$e(r) := e_{r_1} \otimes \cdots \otimes e_{r_n} \in \otimes^n \mathcal{Z}_0 \,.$$

Remark that $\{e(r)\}_{r \in \mathbb{N}^n}$ is an O.N.B of $\otimes^n \mathcal{Z}_0$ and for any $\Phi^{(n)} \in \otimes^{alg,n} Q(A)$ one can write $\Phi^{(n)} = \sum_{r \in \mathbb{N}^n} \lambda(r) e(r)$ (we may assume without loss of generality that the sum is finite). Hence

$$\begin{aligned}
|q_{1,2}^{(n)}(\Phi^{(n)}, \Phi^{(n)})| &= |\sum_{r,s \in \mathbb{N}^n} \overline{\lambda(r)} \lambda(s)\; q_{1,2}^{(n)}\big(e(r), e(s)\big)| \\
&\leqslant \left| \sum_{r_3, \cdots, r_n} q\left(\sum_{r_1, r_2} \lambda(r_1, r_2, r_3, \cdots, r_n) e_{r_1} \otimes e_{r_2} \,;\, \sum_{s_1, s_2} \lambda(s_1, s_2, r_3, \cdots, r_n) e_{s_1} \otimes e_{s_2} \right) \right| \\
&\leqslant a \sum_{r_3, \cdots, r_n} \Big\langle \sum_{r_1, r_2} \lambda(r_1, r_2, r_3, \cdots, r_n) e_{r_1} \otimes e_{r_2} ; A_1 + A_2 \sum_{s_1, s_2} \lambda(s_1, s_2, r_3, \cdots, r_n) e_{s_1} \otimes e_{s_2} \Big\rangle \\
&\quad + b \sum_{r_1, r_2} |\lambda(r_1, r_2, r_3, \cdots, r_n)|^2 \\
&\leqslant a\langle \Phi^{(n)}, A_1 + A_2\, \Phi^{(n)} \rangle + b\, \|\Phi^{(n)}\|^2_{\otimes^n \mathcal{Z}_0} \,.
\end{aligned}$$

The second inequality follows using (**A2**). ∎

Remark 5.3.2. *A consequence of the last proof is that for any $\Psi^{(N)}, \Phi^{(N)} \in \vee^{alg,N} Q(A) = \mathcal{S}_N \otimes^{alg,N} Q(A)$,*

$$q_{i,j}^{(N)}(\Psi^{(N)}, \Phi^{(N)}) = q_{1,2}^{(N)}(\Psi^{(N)}, \Phi^{(N)})\,.$$

Lemma 5.3.3. *Assume* (**A1**)-(**A2**)*. Then q_N extends to a symmetric quadratic form on $Q(H_N^0) \subset \vee^N \mathcal{Z}$. Moreover, for any $\Psi^{(N)} \in Q(H_N^0)$,*

$$|q_N(\Psi^{(N)}, \Psi^{(N)})| \leqslant a\,\langle \Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle + bN\|\Psi^{(N)}\|^2_{\vee^N \mathcal{Z}_0}\,. \tag{5.3.3}$$

Proof. As in the previous lemma, it is enough to prove the inequality (5.3.3) for any $\Psi \in \vee^{alg,N} Q(A)$. Lemma 5.3.1 with Remark 5.3.2 yield the estimate:

$$\begin{aligned} |q_N(\Psi^{(N)}, \Psi^{(N)})| &= \frac{N(N-1)}{2N} |q_{1,2}^{(N)}(\Psi^{(N)}, \Psi^{(N)})| \\ &\leqslant \frac{N}{2}\left[a\langle \Psi^{(N)}, A_1 + A_2 \Psi^{(N)}\rangle + b\|\Psi^{(N)}\|^2_{\vee^N \mathcal{Z}_0}\right]. \end{aligned}$$

Using the fact that $\langle \Psi^{(N)}, A_1 + A_2 \Psi^{(N)}\rangle = \frac{2}{N}\langle \Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle$, we obtain the claimed inequality. ∎

The lemma above allows to use the KLMN Theorem [101, Theorem X.17] since q_N is a small perturbation in the sense of quadratic forms of H_N^0 and therefore one obtains the selfadjointness of H_N.

Proposition 5.3.4 (Self-adjoint realization of H_N)**.** *Assume* (**A1**)-(**A2**)*, then there exists a unique self-adjoint operator H_N with $Q(H_N) = Q(H_N^0)$ satisfying for any $\Psi^{(N)}, \Phi^{(N)} \in Q(H_N^0)$*

$$\langle \Psi^{(N)}, H_N \Phi^{(N)}\rangle = \langle \Psi^{(N)}, H_N^0 \Phi^{(N)}\rangle + q_N(\Psi^{(N)}, \Phi^{(N)}).$$

5.3.2 Invariance property

A straightforward consequence of Proposition 5.3.4 is that the form domain $Q(H_N^0)$ is invariant with respect to the dynamics of H_N. However, we would like to have a quantitative uniform bound on $\langle \Psi_t^{(N)}, H_N^0 \Psi_t^{(N)}\rangle$ for every $t \in \mathbb{R}$. Here

$$\Psi_t^{(N)} := e^{-itH_N} \Psi^{(N)}\,.$$

Proposition 5.3.5 (Propagation of states on $Q(H_N^0)$)**.** *Let $\Psi^{(N)} \in Q(H_N^0)$ such that $\|\Psi^{(N)}\|_{\vee^N \mathcal{Z}_0} = 1$ and satisfying:*

$$\exists C > 0,\ \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle \leqslant CN.$$

Then there exists a constant $C_{a,b} > 0$ independent of N such that for any $t \in \mathbb{R}$ and $N \in \mathbb{N}$,

$$\langle \Psi_t^{(N)}, H_N^0 \Psi_t^{(N)}\rangle \leqslant C_{a,b} N.$$

Proof. Since $0 < a < 1$ the inequality $\pm q_N \leqslant aH_N^0 + bN$ implies that $H_N^0 \leqslant \frac{1}{1-a} H_N + \frac{b}{1-a} N$ in the form sense. Let $\Psi^{(N)} \in Q(H_N^0)$ then for any $t \in \mathbb{R}$,

$$\begin{aligned}
\langle \Psi_t^{(N)}, H_N^0 \Psi_t^{(N)} \rangle &\leqslant \frac{1}{1-a} \langle \Psi_t^{(N)}, H_N \Psi_t^{(N)} \rangle + \frac{b}{1-a} N \\
&\leqslant \frac{1+a}{1-a} \langle \Psi^{(N)}, H_N^0 \Psi^{(N)} \rangle + \frac{2b}{1-a} N \\
&\leqslant \frac{(1+a)C + 2b}{1-a} N \,.
\end{aligned}$$

The second inequality follows using the fact that $\langle \Psi_t^{(N)}, H_N \Psi_t^{(N)} \rangle = \langle \Psi^{(N)}, H_N \Psi^{(N)} \rangle$ and Lemma 5.3.3. ∎

5.4 Duhamel's formula

The main result provided by Theorem 5.2.2 is the identification of the Wigner measures of time-evolved states $\varrho_N(t)$. According to the Definition 5.2.1 of Wigner measures one needs simply to compute the limit when $N \to \infty$ of

$$\mathcal{I}_N(t) := \operatorname{Tr}\left[\varrho_N(t)\, \mathcal{W}(\sqrt{2}\pi\xi)\right] = \langle \Psi_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi) \Psi_t^{(N)} \rangle \,.$$

This task may seems quite simple but since the quantum dynamics are non trivial it is unlikely that one can compute explicitly the above limits. Therefore it seems reasonable to rely on the dynamical properties of $\mathcal{I}_N(t)$ as for non-homogenous PDE and write a Duhamel's formula satisfied by $\mathcal{I}_N(t)$. The point here is that all the possible limits of $\mathcal{I}_N(t)$ have to satisfy a limiting integral equation. And if one can solve the latter equation then it is possible to identify the Wigner measures of $\varrho_N(t)$. This strategy was introduced in [12] for Schrödinger dynamics with singular potential. Here we improve it and extend it to a more general setting.

5.4.1 Commutator computation

In order to derive the aforementioned Duhamel's formula, we differentiate the quantity $\mathcal{I}_N(t)$ with respect to time. This roughly leads to the analysis of the commutator $[\mathcal{W}(\sqrt{2}\pi\xi), H_N - H_N^0]$. Since the Weyl operator do not conserve the number of particles the latter quantity has to be expanded in the symmetric Fock space. To handle this computation efficiently we use the Wick quantization procedure explained in Section 2.1.3 and rely particularly in the properties of the class of symbols $\mathcal{Q}_{p,q}(A)$. We suggest the reading of Section 2.1.4 before going through this subsection.
Recall that $\mathfrak{Q}_n = Q(H_n^0)$ is a Hilbert space equipped with the inner product (2.1.13). The class of monomials $\mathcal{Q}_{p,q}(A)$ is defined by (2.1.14) and the energy functional satisfies:

$$h(z) = \langle z, Az \rangle + \frac{1}{2} q(z^{\otimes 2}, z^{\otimes 2}) \in Q_{1,1}(A) + Q_{2,2}(A) \,,$$

with the following relation holding for all $\Psi^{(N)}, \Phi^{(N)} \in Q(H_N^0)$,

$$\langle \Psi^{(N)}, H_N \Phi^{(N)} \rangle = \langle \Psi^{(N)}, \varepsilon^{-1} h^{Wick} \Phi^{(N)} \rangle , \quad \text{when} \quad \varepsilon = \frac{1}{N} .$$

The above identity stresses the relationship between the many-body Hamiltonian H_N and the Wick quantization of the energy functional $h(z)$. It allows to exploit the general properties of Wick calculus while we deal with the dynamics of H_N.
We define the following monomial q_s for any $z \in Q(A)$, $s \in \mathbb{R}$,

$$q_s(z) := \frac{1}{2} q\big((e^{-isA} z)^{\otimes 2}, (e^{-isA} z)^{\otimes 2}\big) = \frac{1}{2} \langle (e^{-isA} z)^{\otimes 2}, \tilde{q}\, (e^{-isA} z)^{\otimes 2} \rangle , \tag{5.4.1}$$

and check that under the assumption (**A2**),

$$q_s \in \mathcal{Q}_{2,2}(A) \quad \text{with} \quad \tilde{q}_s = \frac{1}{2} e^{isA} \otimes e^{isA} \mathcal{S}_2\, \tilde{q} \mathcal{S}_2\, e^{-isA} \otimes e^{-isA} \in \mathcal{L}(\mathfrak{Q}_2, \mathfrak{Q}_2') .$$

A simple computation yields for any $z \in Q(A)$ and $\xi \in Q(A)$,

$$q_s(z + i\varepsilon\pi\xi) - q_s(z) = \sum_{j=1}^{4} \varepsilon^{j-1} q_j(\xi, s) ,$$

with the monomials $(q_j(\xi, s)[z])_{j=1,2,3,4}$ defined by:

$$\begin{aligned}
&q_1(\xi, s)[z] = -\pi \ \mathrm{Im}\ q(z_s^{\otimes 2}, \mathcal{S}_2\, \xi_s \otimes z_s) , && q_2(\xi, s)[z] = -\frac{\pi^2}{2} \mathrm{Re}\ q(z_s^{\otimes 2}, \xi_s^{\otimes 2}) + 2\pi^2 q(\mathcal{S}_2 \xi_s \otimes z_s, \mathcal{S}_2 \xi_s \otimes z_s) , \\
&q_3(\xi, s)[z] = \pi^3\, \mathrm{Im}\ q(\xi_s^{\otimes 2}, \mathcal{S}_2\, \xi_s \otimes z_s) , && q_4(\xi, s)[z] = \frac{\pi^4}{4} q(\xi_s^{\otimes 2}, \xi_s^{\otimes 2}),
\end{aligned} \tag{5.4.2}$$

and the notation:

$$\xi_s := e^{-isA} \xi , \qquad z_s := e^{-isA} z .$$

Lemma 5.4.1. *Assume* (**A1**)-(**A2**)*, then one checks that*

$$\begin{aligned}
&q_1(\xi, s)[z] \in \mathcal{Q}_{2,1}(A) + \mathcal{Q}_{1,2}(A) , && q_2(\xi, s)[z] \in \mathcal{Q}_{2,0}(A) + \mathcal{Q}_{0,2}(A) + \mathcal{Q}_{1,1}(A) , \\
&q_3(\xi, s)[z] \in \mathcal{Q}_{1,0}(A) + \mathcal{Q}_{0,1}(A) , && q_4(\xi, s)[z] \in \mathcal{Q}_{0,0}(A).
\end{aligned}$$

Proof. This result is a straightforward consequence of Proposition 2.1.30 (iv). However for reader convenience we provide a direct proof. Remark that $q_1(\xi, s)[z]$ is a linear combination of two conjugate monomials. So it is enough to check that $q(z^{\otimes 2}, \xi \otimes z) \in \mathcal{Q}_{1,2}(A)$. In fact, we have

$$\begin{aligned}
b(z) = q(z^{\otimes 2}, \xi \otimes z) \ &= \ \langle z^{\otimes 2}, \mathcal{S}_2\, \tilde{q}\, \xi \otimes z \rangle \\
&= \ \langle z^{\otimes 2}, \mathcal{S}_2\, \tilde{q}\, (|\xi\rangle \otimes 1)\, z \rangle .
\end{aligned}$$

This implies that there exists a unique operator $\tilde{b} = \mathcal{S}_2\, \tilde{q}\, |\xi\rangle \otimes 1$ such that for any $z \in Q(A)$,

$$b(z) = \langle z^{\otimes 2}, \tilde{b}\, z \rangle .$$

Moreover $\tilde{b} \in \mathcal{L}(\mathfrak{Q}_1, \mathfrak{Q}_2')$ (here $\mathfrak{Q}_n = Q(H_n^0)$) since $\xi \in Q(A)$ and

$$\left((A_1 + A_2 + 1)^{-\frac{1}{2}} \tilde{q}\, (A+1)^{-\frac{1}{2}} \otimes (A+1)^{-\frac{1}{2}}\right) |(A+1)^{\frac{1}{2}}\xi\rangle \otimes 1 \quad \in \mathcal{L}(\mathcal{Z}_0, \otimes^2 \mathcal{Z}_0)\,.$$

Hence $b \in \mathcal{Q}_{1,2}(A)$ and $\bar{b} \in \mathcal{Q}_{2,1}(A)$ according to Proposition (2.1.30) (i). ∎

Proposition 5.4.2. *For $\xi \in Q(A)$ and $\varepsilon = \frac{1}{N}$, we have the following equality in the sense of quadratic forms on $Q(H_N^0)$,*

$$\frac{1}{\varepsilon}\left[q_s^{Wick}, \mathcal{W}(\sqrt{2}\pi\xi)\right] = \mathcal{W}(\sqrt{2}\pi\xi)\Big[\sum_{j=1}^{4} \varepsilon^{j-1} q_j(\xi, s)^{Wick}\Big], \tag{5.4.3}$$

where $q_j(\xi, s)$, $j = 1, 2, 3, 4$, are the monomials defined in (5.4.2) *and q_s is given by* (5.4.1).

Proof. This follows by applying Proposition 2.1.30 (v). ∎

5.4.2 Integral equation

Let $(\Psi^{(N)})_{N\in\mathbb{N}}$ be a sequence of normalized vectors in $Q(H_N^0) \subset \bigvee^N \mathcal{Z}_0$ satisfying the hypothesis of Theorem 5.2.2. The time evolved state is

$$\varrho_N(t) := |\Psi_t^{(N)}\rangle\langle\Psi_t^{(N)}| \quad \text{where} \quad \Psi_t^{(N)} := e^{-itH_N}\Psi^{(N)}\,.$$

Actually, it is convenient to work within the interaction representation

$$\widetilde{\varrho}_N(t) := |\widetilde{\Psi}_t^{(N)}\rangle\langle\widetilde{\Psi}_t^{(N)}| \quad \text{where} \quad \widetilde{\Psi}_t^{(N)} := e^{itH_N^0} e^{-itH_N}\Psi^{(N)}\,. \tag{5.4.4}$$

Our aim in this subsection is to write an integral equation (or Duhamel's formula) satisfied by the map

$$t \mapsto \mathcal{J}_N(t) := \operatorname{Tr}\left[\widetilde{\varrho}_N(t)\, \mathcal{W}(\sqrt{2}\pi\xi)\right] = \langle \widetilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\, \widetilde{\Psi}_t^{(N)}\rangle\,, \tag{5.4.5}$$

and to put it in a convenient form in order to carry on the limit $N \to \infty$.

Proposition 5.4.3. *Assume* **(A1)**-**(A2)** *and consider a sequence $(\Psi^{(N)})_{N\in\mathbb{N}}$ of normalized vectors in $Q(H_N^0)$. Then for any $\xi \in D(A)$ the map $t \in \mathbb{R} \mapsto \mathcal{J}_N(t)$ defined in* (5.4.5) *is C^1 and satisfies for $\varepsilon = \frac{1}{N}$ and all $t \in \mathbb{R}$,*

$$\mathcal{J}_N(t) = \mathcal{J}_N(0) + i\int_0^t \Big\langle \widetilde{\Psi}_s^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\left[\sum_{j=1}^{4} \varepsilon^{j-1}\Big(q_j(\xi, s)\Big)^{Wick}\right]\widetilde{\Psi}_s^{(N)}\Big\rangle\, ds, \tag{5.4.6}$$

where $q_j(\xi, s)$, $j = 1, \cdots, 4$, are the monomials given in (5.4.2).

Proof. By Stone's theorem one can see that $\mathcal{J}_N(t)$ is continuously differentiable since $\Psi^{(N)} \in Q(H_N) = Q(H_N^0)$. So one obtains

$$i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_N(t) = \langle \widetilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)e^{itH_N^0}(H_N - H_N^0)e^{-itH_N}\Psi^{(N)}\rangle - \langle e^{itH_N^0}(H_N - H_N^0)e^{-itH_N}\Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\widetilde{\Psi}_t^{(N)}\rangle .$$

Using the fact that $\varepsilon^{-1}q^{Wick}_{|\vee^N\mathcal{Z}_0} = H_N - H_N^0 = q_N$ in the sense of quadratic forms on $Q(H_N^0)$ and Proposition 2.1.30, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_N(t) &= \langle -\frac{i}{\varepsilon}e^{itH_N^0}q^{Wick}e^{-itH_N}\Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\tilde{\Psi}_t^{(N)}\rangle + \langle \tilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi) - \frac{i}{\varepsilon}e^{itH_N^0}q^{Wick}e^{-itH_N}\Psi^{(N)}\rangle \\ &= \frac{i}{\varepsilon}\langle \widetilde{\Psi}_t^{(N)}, \left[q_t^{Wick}, \mathcal{W}(\sqrt{2}\pi\xi)\right]\widetilde{\Psi}_t^{(N)}\rangle ,\end{aligned}$$

where $q_t(z) = \frac{1}{2}q(z_t^{\otimes^2}, z_t^{\otimes^2}) \in \mathcal{Q}_{2,2}(A)$. The commutator and the duality bracket in the last equations make sense since $\mathcal{W}(\sqrt{2}\pi\xi)\widetilde{\Psi}_t^{(N)} \in Q(d\Gamma(A) + \mathbf{N})$ by Proposition 2.1.31. So, the N^{th} component $\left[\mathcal{W}(\sqrt{2}\pi\xi)\widetilde{\Psi}_t^{(N)}\right]^{(N)}$ belongs to $Q(H_N^0)$. Now, we conclude by applying Proposition 5.4.2. ∎

5.5 Convergence arguments

We have established in the previous section an integral equation (5.4.6) satisfied by the quantity $\mathcal{J}_N(t)$. Here we consider its limit when $N \to \infty$. The main steps are the analysis of $\partial_t\mathcal{J}_N(t)$ and the extraction of subsequences $(N_k)_{k\in\mathbb{N}}$ that would lead to a convergent integral equation for all times. This is achieved under the assumptions (**D1**) and (**D2**).

5.5.1 Convergence of $\partial_t\mathcal{J}_N(t)$

The following property is crucial for the proof of convergence.

Proposition 5.5.1. *Let $\{\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}^*}$ be a sequence of normal states on $\vee^N\mathcal{Z}_0$ such that $\mathcal{M}(\varrho_N, N \in \mathbb{N}) = \{\mu\}$ and*

$$\exists C > 0, \forall N \in \mathbb{N}, \langle \Psi^{(N)}, H_N^0\Psi^{(N)}\rangle \leqslant CN . \tag{5.5.1}$$

Assume (**A1**)-(**A2**) *and suppose that either* (**D1**) *or* (**D2**) *is true, then for any $\xi \in Q(A)$ and for every $s \in \mathbb{R}$,*

$$\lim_{\substack{N\to+\infty \\ N\varepsilon=1}} \langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\,[q_1(\xi,s)]^{Wick}\,\Psi^{(N)}\rangle = \int_{\mathcal{Z}_0} e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}q_1(\xi,s)[z]\,d\mu(z) , \tag{5.5.2}$$

where $z_s = e^{-isA}z$, $\xi_s = e^{-isA}z$ and $q_1(\xi,s)[z] = -\pi\ \mathrm{Im}\ q(z_s^{\otimes 2}, \mathcal{S}_2\,\xi_s\otimes z_s)$.

Proof. For simplicity we assume $s = 0$ since the proof goes exactly the same when $s \neq 0$. The following writing holds for any $\xi, z \in Q(A)$,

$$2q_1(\xi,0)[z] = -2\pi\ \mathrm{Im}\ q(z^{\otimes 2}, \mathcal{S}_2\,\xi\otimes z) = i\pi B_1(z) - i\pi B_2(z),$$

with

$$B_1(z) = \langle \xi \otimes z, \mathcal{S}_2\, \tilde{q} z^{\otimes 2} \rangle, \quad B_2(z) = \langle z^{\otimes 2}, \tilde{q}\, \mathcal{S}_2\, (\xi \otimes z) \rangle.$$

By the assumption (**A2**), the two symbols B_1 and B_2 belong to $\mathcal{Q}_{2,1}(A)$ and $\mathcal{Q}_{1,2}(A)$ respectively with

$$\tilde{B}_1 = \langle \xi | \otimes 1\; \mathcal{S}_2\; \tilde{q}\; \mathcal{S}_2 \in \mathcal{L}(\mathfrak{Q}_2, \mathfrak{Q}_1'), \qquad \tilde{B}_2 = \mathcal{S}_2\; \tilde{q}\; \mathcal{S}_2\, |\xi\rangle \otimes 1 \in \mathcal{L}(\mathfrak{Q}_1, \mathfrak{Q}_2'),$$

and for any $z \in Q(A)$, $B_1(z) = \langle z, \tilde{B}_1 z^{\otimes 2} \rangle$ and $B_2(z) = \langle z^{\otimes 2}, \tilde{B}_2 z \rangle$ with the property $\overline{B_1(z)} = B_2(z)$. We will use an approximation argument. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ if $|x| \leqslant 1$, $\chi(x) = 0$ if $|x| \geqslant 2$ and $0 \leqslant \chi \leqslant 1$. We denote for $m \in \mathbb{N}^*$, $\chi_m(x) = \chi(\frac{x}{m})$ and $H_1^0 = A$, $H_2^0 = A_1 + A_2$ and set

$$\tilde{B}_{1,m} := \chi_m(H_1^0)\, \tilde{B}_1\, \chi_m(H_2^0) \in \mathcal{L}(\vee^2 \mathcal{Z}_0, \mathcal{Z}_0), \qquad \tilde{B}_{2,m} := \chi_m(H_2^0)\, \tilde{B}_2\, \chi_m(H_1^0) \in \mathcal{L}(\mathcal{Z}_0, \vee^2 \mathcal{Z}_0),$$

and

$$B_{1,m}(z) = \langle z, \tilde{B}_{1,m} z^{\otimes 2} \rangle, \qquad B_{2,m}(z) = \langle z^{\otimes 2}, \tilde{B}_{2,m} z \rangle.$$

Since (**D1**) says that A has a compact resolvent and both operators $(H_1^0 + 1)^{-\frac{1}{2}} \tilde{B}_1 (H_2^0 + 1)^{-\frac{1}{2}}$ and $(H_2^0 + 1)^{-\frac{1}{2}} \tilde{B}_2 (H_1^0 + 1)^{-\frac{1}{2}}$ are either compact or bounded, we see that $B_{j,m}$ are compact operators once we assume (**D1**) or (**D2**). We now write the following inequalities for $j = 1, 2$,

$$|\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\, B_j^{Wick}\, \Psi^{(N)} \rangle - \mu(e^{2i\pi \operatorname{Re}\langle \xi, z\rangle} B_j(z))| \leqslant \mathcal{A}_j^{(m)} + \mathcal{B}_j^{(m)} + \mathcal{C}_j^{(m)}, \tag{5.5.3}$$

where

$$\mathcal{A}_j^{(m)} = |\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)[B_j - B_{j,m}]^{Wick} \Psi^{(N)} \rangle|,$$

$$\mathcal{B}_j^{(m)} = |\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\, B_{j,m}^{Wick} \Psi^{(N)} \rangle - \mu(e^{2i\pi \operatorname{Re}\langle \xi, z\rangle} B_{j,m}(z))|,$$

and

$$\mathcal{C}_j^{(m)} = |\mu(e^{2i\pi \operatorname{Re}\langle \xi, z\rangle} B_{j,m}(z)) - \mu(e^{2i\pi \operatorname{Re}\langle \xi, z\rangle} B_j(z))|.$$

To prove the limit (5.5.2), we show that all the terms $\mathcal{A}_j^{(m)}, \mathcal{B}_j^{(m)}, \mathcal{C}_j^{(m)}$ can be made arbitrary small for all N larger enough by choosing a convenient $m \in \mathbb{N}$.

The term $\mathcal{C}_j^{(m)}$: By dominated convergence theorem the quantity $\mathcal{C}_j^{(m)}$ tends to 0 when $m \to \infty$ for $j = 1, 2$. In fact $B_{j,m}(z)$ converges to $B_j(z)$ for all $z \in Q(A)$ since $s - \lim \chi_m(H_j^0) = \mathrm{Id}$. Moreover, we have for some $C' > 0$ and any $z \in Q(A)$,

$$|B_{j,m}(z)| \leqslant C' \|\xi\|_{Q(A)}\, \|z\|_{Q(A)}^2\, \|z\|_{\mathcal{Z}_0}, \tag{5.5.4}$$

since $B_{j,m}$ are in $\mathcal{Q}_{1,2}(A)$ or $\mathcal{Q}_{2,1}(A)$ and by Proposition 2.2.22 we get the a priori estimate:

$$\int_{\mathcal{Z}_0} \|z\|_{Q(A)}^2 \|z\|_{\mathcal{Z}_0}\, d\mu(z) \leqslant C. \tag{5.5.5}$$

The term $\mathcal{B}_j^{(m)}$: Since $\tilde{B}_{j,m}$ are compact operators for $j = 1, 2$ and any $m \in \mathbb{N}^*$, the quantity $\mathcal{B}_j^{(m)} \to 0$ when $N \to \infty$ owing to result proved in [9, Theorem 6.13] and [9, Corollary 6.14].

The term $\mathcal{A}_j^{(m)}$: We consider only $j = 1$ since the case $j = 2$ is quite similar. We write for any $z \in Q(A)$,

$$B_1(z) - B_{1,m}(z) = \langle z, (1 - \chi_m(H_1^0))\tilde{B}_1\, z^{\otimes 2}\rangle + \langle z, \chi_m(H_1^0)\tilde{B}_1(1 - \chi_m(H_2^0))\, z^{\otimes 2}\rangle =: \mathcal{U}_1(z) + \mathcal{U}_2(z)\,,$$

and check that $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{Q}_{2,1}(A)$. Let $\Phi^{(N-1)} = [\mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N)}]^{(N-1)}$ be the $(N-1)^{th}$ component of the vector $\mathcal{W}(\sqrt{2}\pi\xi)\Psi^{(N)}$ in the symmetric Fock space $\Gamma_s(\mathcal{Z}_0)$. By Proposition 2.1.31 we see that $\Phi^{(N-1)} \in Q(H_{N-1}^0)$. So, one obtains

$$\mathcal{A}_1^{(m)} = \underbrace{\langle \Phi^{(N-1)}, \mathcal{U}_1^{Wick}\,\Psi^{(N)}\rangle}_{(1)} + \underbrace{\langle \Phi^{(N-1)}, \mathcal{U}_2^{Wick}\,\Psi^{(N)}\rangle}_{(2)}.$$

Now estimate each term. Let denote $\overline{\chi}_m = 1 - \chi_m$ then for $\lambda > 0$ and $\varepsilon = \frac{1}{N}$,

$$\begin{aligned}
\Big|(1)\Big| &= \Big|\langle \Phi^{(N-1)}, \varepsilon^{3/2}\sqrt{N(N-1)^2}\,\mathcal{S}_{N-1}\,\overline{\chi}_m(H_1^0)\tilde{B}_1 \otimes 1^{(N-2)}\,\Psi^{(N)}\rangle\Big| \\
&\leqslant \Big|\langle \overline{\chi}_m(H_1^0) \otimes 1^{(N-2)}\Phi^{(N-1)},\ \tilde{B}_1 \otimes 1^{(N-2)}\,\Psi^{(N)}\rangle\Big| \\
&\leqslant \alpha(\lambda)\Big\|(H_1^0 + \lambda)^{1/2}\overline{\chi}_m(H_1^0) \otimes 1^{(N-2)}\Phi^{(N-1)}\Big\|\ \Big\|(H_2^0 + 1)^{1/2} \otimes 1^{(N-2)}\Psi^{(N)}\Big\|,
\end{aligned}$$

where

$$\alpha(\lambda) = \Big\|(H_1^0 + \lambda)^{-1/2}\tilde{B}_1(H_2^0 + 1)^{-1/2}\Big\|_{\mathcal{L}(\bigvee^2 \mathcal{Z}_0, \mathcal{Z}_0)} \to 0, \quad \text{when } \lambda \to \infty\,.$$

Remark that the spectral theorem yields,

$$\forall m \in \mathbb{N}^*,\ \|\overline{\chi}_m(A)\ (A+1)^{-\frac{1}{2}}\|^2_{\mathcal{L}(\mathcal{Z}_0)} \leqslant \frac{1}{m}.$$

So using the assumption (5.5.1), the symmetry of $\Phi^{(N-1)}$ and Proposition 2.1.31, one obtains

$$\Big\|(H_1^0 + \lambda)^{1/2}\overline{\chi}_m(H_1^0) \otimes 1^{(N-2)}\Phi^{(N-1)}\Big\| \leqslant C_1\sqrt{1 + \frac{\lambda}{m}}\,,$$

form some $C_1 > 0$ independent of N. Hence $|(1)| \lesssim \alpha(\lambda)\sqrt{1 + \frac{\lambda}{m}}$ and if we choose $\lambda = m$ we see that $|(1)| \to 0$ when $m \to \infty$.
Similar computation yields for λ large enough

$$\Big|(2)\Big| \leqslant \beta(\lambda)\Big\|(H_1^0 + \lambda)^{1/2} \otimes 1^{(N-2)}\Phi^{(N-1)}\Big\|\ \Big\|\overline{\chi}_m(H_2^0)(H_2^0 + \lambda)^{1/2} \otimes 1^{(N-2)}\Psi^{(N)}\Big\|,$$

where

$$\beta(\lambda) = \Big\|(H_1^0 + 1)^{-1/2}\tilde{B}_1(H_2^0 + \lambda)^{-1/2}\Big\|_{\mathcal{L}(\vee^2 \mathcal{Z}_0, \mathcal{Z}_0)} \to 0, \quad \text{when } \lambda \to \infty\,.$$

So by the same argument above we conclude that $|(2)| \lesssim \beta(\lambda)\sqrt{1+\frac{\lambda}{m}}$ and if we choose again $\lambda = m$ we get $|(2)| \to 0$ when $m \to \infty$.
This proves the claimed limit (5.5.2) for any $\xi \in D \subset \mathcal{Z}_0$. So we extend this result to any $\xi \in Q(A)$ by an approximation argument. In fact take for any $\xi \in Q(A)$ a sequence $(\xi_m)_{m\in\mathbb{N}}$ such that $\xi_m \to \xi$ in $Q(A)$. Write

$$\left|\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\,[q_1(\xi,0)]^{Wick}\,\Psi^{(N)}\rangle - \int_{\mathcal{Z}_0} e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle} q_1(\xi,0)[z]\,d\mu(z)\right| \;\leqslant\; \mathcal{A}^{(m)} + \mathcal{B}^{(m)} + \mathcal{C}^{(m)},$$

with

$$\mathcal{A}^{(m)} = \left|\langle \Psi^{(N)}, \Big(\mathcal{W}(\sqrt{2}\pi\xi) - W(\sqrt{2}\pi\xi_m)\Big) q_1(\xi,0)^{Wick}\Psi^{(N)}\rangle\right|,$$

$$\mathcal{B}^{(m)} = \left|\langle \Psi^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi_m)\, q_1(\xi,0)^{Wick}\Psi^{(N)}\rangle - \mu(e^{2i\pi\,\mathrm{Re}\,\langle\xi_m,z\rangle} q_1(\xi,0)[z])\right|,$$

and

$$\mathcal{C}^{(m)} = \left|\mu(e^{2i\pi\,\mathrm{Re}\,\langle\xi_m,z\rangle} q_1(\xi,0)[z]) - \mu(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle} q_1(\xi,0)[z])\right|.$$

So using Number-Weyl estimates in [9, Lemma 3.1], one shows that $\mathcal{A}^{(m)} \lesssim ||\xi - \xi_m||_{\mathcal{Z}_0}$ and hence $\mathcal{A}^{(m)} \to 0$. Now, $\mathcal{B}^{(m)} \to 0$ by the result proved above and $\mathcal{C}^{(m)} \to 0$ by (5.5.4)-(5.5.5) and the dominated convergence theorem. ∎

5.5.2 Existence of Wigner measures for all times

Wigner measures and their properties were studied in infinite dimensional spaces in [9]. A result proved in [9, Theorem 6.2] says that for any sequence of normal states $\{\widetilde{\varrho}_N(t)\}_{N\in\mathbb{N}}$ as in (5.4.4) we can extract a subsequence $(N_k)_{k\in\mathbb{N}}$ such that $\widetilde{\varrho}_{N_k}(t)$ has a unique Wigner measure $\tilde{\mu}_t$ according to Definition 5.2.1. However, the subsequence may depend in the time $t \in \mathbb{R}$. So, in order to carry on the limit on the integral equation (5.4.6) we need to extract a subsequence $(N_k)_{k\in\mathbb{N}}$ for all $t \in \mathbb{R}$ that gives $\mathcal{M}(\widetilde{\varrho}_{N_k}(t), k \in \mathbb{N}) = \{\tilde{\mu}_t\}$.

Proposition 5.5.2. *Let $\{\varrho_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ be a sequence of normal states on $\vee^N \mathcal{Z}_0$ such that*

$$\exists C > 0, \forall N \in \mathbb{N},\ \langle \Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle \leqslant CN\,,$$

and $\mathcal{M}(\varrho_N, N \in \mathbb{N}) = \{\mu_0\}$. Then for any $\xi \in Q(A)$ and for any subsequence $(N_k)_{k\in\mathbb{N}}$ there exist a family of probability measures $(\mu_t)_{t\in\mathbb{R}}$ on $\mathcal{Z}_0$ and a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ such that for all $t \in \mathbb{R}$,

$$\mathcal{M}\left(\left|e^{-itH^0_{N_{k_l}}} e^{itH_{N_{k_l}}}\Psi^{(N_{k_l})}\rangle\langle e^{-itH^0_{N_{k_l}}} e^{itH_{N_{k_l}}}\Psi^{(N_{k_l})}\right|, l \in \mathbb{N}\right) = \{\tilde{\mu}_t\},$$

and the following Liouville equation is satisfied for any $\xi \in Q(A)$,

$$\begin{aligned}\tilde{\mu}_t(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}) &= \tilde{\mu}_0(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}) + i\int_0^t \tilde{\mu}_s(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle} q_1(\xi,s)[z])\,ds \\ &= \tilde{\mu}_0(e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}) + i\int_0^t \tilde{\mu}_s\big(\{q_s(z); e^{2i\pi\,\mathrm{Re}\,\langle\xi,z\rangle}\}\big)\,ds,\end{aligned} \tag{5.5.6}$$

with $z_s = e^{-isA}z$, $\xi_s = e^{-isA}\xi$, $q_1(\xi, s) = -\pi \ \mathrm{Im}\ q(z_s^{\otimes 2}, \mathcal{S}_2\, \xi_s \otimes z_s)$, $q_s(z) = \frac{1}{2}q(z_s^{\otimes 2}, z_s^{\otimes 2})$ *and the bracket* $\{b_1(z); b_2(z)\}$ *equals to* $\partial_{\bar z} b_1(z) \cdot \partial_z b_2(z) - \partial_{\bar z} b_2(z) \cdot \partial_z b_1(z)$.

Proof. The extraction of such subsequence $(N_{k_l})_{l\in\mathbb{N}}$ for all times follows by an Ascoli type argument proved in [11, Proposition 3.3]. Here we briefly check the main points. Wigner measures are identified through (5.2.8). Hence we consider the quantities:

$$G_N(t,\xi) = \langle \widetilde{\Psi}_t^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi)\, \widetilde{\Psi}_t^{(N)}\rangle.$$

We wish to prove the existence of a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ such that $G_{N_{k_l}}(t,\xi)$ converges for all $t \in \mathbb{R}$ and $\xi \in \mathcal{Z}_0$. For this, we exploit the regularity of the functions $G_N(t,\xi)$ with respect to t and ξ. In some sense we have to prove that the family $(G_N)_{N\in\mathbb{N}}$ is equi-continuous on bounded sets of $\mathbb{R} \times \mathcal{Z}_0$. By using Lemma 3.1 in [9] we get for $\xi, \eta \in Q(A)$,

$$\|\big[\mathcal{W}(\sqrt{2}\pi\xi) - \mathcal{W}(\sqrt{2}\pi\eta)\big](\mathbf{N}+1)^{-\frac{1}{2}}\|_{\mathcal{L}(\Gamma_s(\mathcal{Z}_0))} \lesssim \|\xi - \eta\|_{\mathcal{Z}_0} \sqrt{\|\xi\|_{\mathcal{Z}_0}^2 + \|\eta\|_{\mathcal{Z}_0}^2 + 1}\,.$$

Therefore, the following estimate holds

$$|G_N(t,\xi) - G_N(t,\eta)| \lesssim \|\xi - \eta\|_{\mathcal{Z}_0} \sqrt{\|\xi\|_{\mathcal{Z}_0}^2 + \|\eta\|_{\mathcal{Z}_0}^2 + 1}\,. \tag{5.5.7}$$

On the other hand by using Proposition 5.4.3, Proposition 2.1.30 (iii) and Proposition 2.1.31, we get for any $s, t \in \mathbb{R}, \xi \in Q(A)$ and $\varepsilon = \frac{1}{N}$,

$$\begin{aligned}|G_N(s,\xi) - G_N(t,\xi)| \leqslant &\Big| \int_s^t \langle \widetilde{\Psi}_r^{(N)}, \mathcal{W}(\sqrt{2}\pi\xi) \sum_{j=1}^4 \varepsilon^{j-1}\, q_j(\xi, r)^{Wick}\ \widetilde{\Psi}_r^{(N)}\rangle\, dr \Big| \\ \lesssim &(1 + \|\xi\|_{Q(A)}^4)\ |s-t| \sup_{s\leqslant r\leqslant t} \|(A_1+1)^{\frac{1}{2}}\widetilde{\Psi}_r^{(N)}\|_{\bigvee^N \mathcal{Z}_0}^2 \ \lesssim\ (1 + \|\xi\|_{Q(A)}^4)\,|s-t|\,.\end{aligned}$$

Hence combining (5.5.7) with the latter inequality one gets for any $\eta, \xi \in Q(A)$ and $s, t \in \mathbb{R}$,

$$|G_N(t,\xi) - G_N(s,\eta)| \lesssim |s-t|(1 + \|\xi\|_{Q(A)}^4) + \|\xi - \eta\|_{\mathcal{Z}_0} \sqrt{\|\xi\|_{\mathcal{Z}_0}^2 + \|\eta\|_{\mathcal{Z}_0}^2 + 1}\,.$$

Furthermore the uniform estimate $|G_N(t,\xi)| \leqslant 1$ holds true. By an Ascoli type argument as in [11, Proposition 3.3] and [12, Proposition 3.9], we see that for any sequence $(N_k)_{k\in\mathbb{N}}$, there exists a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ and a family of Borel probability measures $(\tilde\mu_t)_{t\in\mathbb{R}}$ on $\mathcal{Z}_0$ satisfying for any $t \in \mathbb{R}$,

$$\mathcal{M}\left(|\widetilde{\Psi}_t^{(N_{k_l})}\rangle\langle \widetilde{\Psi}_t^{(N_{k_l})}|,\ l \in \mathbb{N}\right) = \{\tilde\mu_t\}\,.$$

Now to prove the integral equation (5.5.6), we use Proposition 5.4.3 with $\varepsilon = \frac{1}{N_{k_l}}$,

$$\mathcal{J}_{N_{k_l}}(t) = \mathcal{J}_{N_{k_l}}(0) + i\int_0^t \langle \widetilde{\Psi}_s^{(N_{k_l})}, W(\sqrt{2}\pi\xi)\big[\sum_{j=1}^4 (\varepsilon^{j-1} q_j(\xi, s)^{Wick}\big]\,\widetilde{\Psi}_s^{(N_{k_l})}\rangle\, ds, \tag{5.5.8}$$

with the monomials $(q_j(\xi,s))_{j=1,2,3,4}$ given by (5.4.2). The estimates provided by Proposition 2.1.30 (iii) and Proposition 2.1.31 give the convergence towards 0 of the terms involving $q_j(\xi,s)^{Wick}$, $j = 2,3,4$ when $l \to \infty$. Applying the Proposition 5.5.1 to the subsequence $\big|\widetilde{\Psi}_s^{(N_{k_l})}\big\rangle\big\langle\widetilde{\Psi}_s^{(N_{k_l})}\big|$, we obtain the claimed equation (5.5.6). Remark that in order to check the hypothesis (5.5.1) of Proposition 5.5.1 we have used Proposition 5.3.5. ∎

5.6 The Liouville equation

Once Proposition 5.5.1 is proved we are led to the problem of solving a Liouville (continuity or transport) equation in infinite dimension which already admits measure-valued solutions. So the point is to prove uniqueness. The method we use for uniqueness here is introduced in [12] and uses some techniques from optimal transport theory initiated in the book [5].

5.6.1 Properties of measure-valued solutions to Liouville equation

We need some preliminaries. The sets of all Borel probability measures on $\mathcal{Z}_0$ and $Q(A)$ are denoted by $\mathfrak{P}(\mathcal{Z}_0)$ and $\mathfrak{P}(Q(A))$ respectively. We introduce some classes of cylindrical functions on $Q(A)$. Denote $\mathbb{P}$ the space of finite rank orthogonal projections on $Q(A)$. We say that a function f is in the cylindrical Schwartz space $\mathcal{S}_{cyl}(Q(A))$ (resp. $C^\infty_{0,cyl}(Q(A))$) if:

$$\exists \mathfrak{p} \in \mathbb{P},\ \exists g \in \mathcal{S}(\mathfrak{p}Q(A))\ \ \big(\textit{resp. } C^\infty_{0,cyl}(\mathfrak{p}Q(A))\big),\ \forall z \in Q(A), f(z) = g(\mathfrak{p}z).$$

The space $C^\infty_{0,cyl}(\mathbb{R}\times Q(A))$ of smooth cylindrical functions with compact support on $\mathbb{R}\times Q(A)$ will be useful too and it is defined in the same way. Denote $L_{\mathfrak{p}}(dz)$ the Lebesgue measure on the finite dimensional subspace $\mathfrak{p}Q(A)$. The Fourier transform of functions in $\mathcal{S}_{cyl}(Q(A))$ are given by

$$\mathcal{F}[f](\xi) = \int_{\mathfrak{p}Q(A)} f(z) e^{-2i\pi \operatorname{Re}\langle z,\xi\rangle_{Q(A)}} L_{\mathfrak{p}}(dz),$$

After fixing a Hilbert basis $(e_n)_{n\in\mathbb{N}^*}$, the space $Q(A)$ can be equipped with a distance,

$$d_w(x_1 - x_2) = \sqrt{\sum_{n\in\mathbb{N}^*} \frac{|\langle x_1 - x_2, e_n\rangle_{Q(A)}|^2}{n^2}}.$$

It induces a topology globally weaker than the weak topology. However these two topologies coincide on bounded sets of $Q(A)$.
The norm and d_w topology lead to two distinct notions of narrow convergence of probability measures. On the one hand, a sequence $(\mu_n)_{n\in\mathbb{N}}$ is narrowly convergent to $\mu \in \mathfrak{P}(Q(A))$ if

$$\lim_{n\to+\infty} \int_{Q(A)} f(z) d\mu_n(z) = \int_{Q(A)} f(z) d\mu(z), \tag{5.6.1}$$

for every function $f \in \mathcal{C}^0_b(Q(A), \|.\|_{Q(A)})$, the space of continuous and bounded real functions defined on $Q(A)$ with the norm topology. On the other hand, a sequence $(\mu_n)_{n\in\mathbb{N}}$ is weakly narrowly convergent if the limit (5.6.1) holds for all $f \in \mathcal{C}^0_b(Q(A), d_w)$.

The family of probability measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ provided by Proposition 5.5.2 have uniformly bounded moments $\int_{Q(A)} \|z\|_{\mathcal{Z}_0}^{2k} d\tilde{\mu}_t(z) \leqslant 1$ for all $k \in \mathbb{N}$ thanks to Proposition 2.2.22. With this property, the weak narrow continuity of the map $t \to \tilde{\mu}_t$ can be checked according to the convergence (5.6.1) for all $f \in \mathcal{S}_{cyl}(Q(A))$ or for all $f \in \mathcal{C}_{0,cyl}^{\infty}(Q(A))$ (this is proved in [5, Lemma 5.1.12 f)]).

Remarks 5.6.1. *The notion of weakly narrowly convergence on $\mathfrak{P}(\mathcal{Z}_0)$ can be defined similarly by replacing $Q(A)$ by $\mathcal{Z}_0$ in the previous preliminary and replacing the distance d_w by $d_{w,\mathcal{Z}_0}$ defined as follows:*

$$d_{w,\mathcal{Z}_0}(x_1 - x_2) = \sqrt{\sum_{n\in\mathbb{N}^*} \frac{|\langle x_1 - x_2, f_n\rangle_{\mathcal{Z}_0}|^2}{n^2}},$$

where $(f_n)_{n\in\mathbb{N}^*}$ is a Hilbert basis of $\mathcal{Z}_0$.

Proposition 5.6.2. *Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ a sequence of normal states in $\bigvee^N \mathcal{Z}_0$ satisfying the uniform estimate:*

$$\exists C > 0\,, \forall N \in \mathbb{N},\ \langle\Psi^{(N)}, H_N^0 \Psi^{(N)}\rangle \leqslant CN\,.$$

Consider an extracted subsequence $(N_k)_{k\in\mathbb{N}}$ according to Proposition 5.5.2 such that for any $t \in \mathbb{R}$,

$$\mathcal{M}(|\widetilde{\Psi}_t^{(N_k)}\rangle\langle\widetilde{\Psi}_t^{(N_k)}|, k \in \mathbb{N}) = \{\tilde{\mu}_t\}\,,$$

where $\widetilde{\Psi}_t^{(N_k)}$ is given by (5.4.4)*. Then the Borel probability measures $\tilde{\mu}_t$ on $\mathcal{Z}_0$ satisfy:*

(i) $\tilde{\mu}_t$ are Borel probability measures on $Q(A)$.

(ii) The map $t \mapsto \tilde{\mu}_t$ is weakly narrowly continuous on $Q(A)$ and $\mathcal{Z}_0$.

(iii) The measure $\tilde{\mu}_t$ is a weak solution to the Liouville equation

$$\partial_t \tilde{\mu}_t + i\{q_t(z); \tilde{\mu}_t\} = 0,$$

i.e: for all $f \in C_{0,cyl}^{\infty}(\mathbb{R} \times Q(A))$

$$\int_{\mathbb{R}} \int_{Q(A)} (\partial_t f(t,z) + i\{q_t(z), f(t,z)\})\, d\tilde{\mu}_t(z)\, dt = 0\,,$$

where $z_t = e^{itA} z$ and $q_t(z) = \frac{1}{2} q(z_t^{\otimes 2}, z_t^{\otimes 2})$.

Proof. The statement (i) is proved in [12, Proposition 3.11] when $A = -\Delta$ but the proof works without any change for a general operator A satisfying (**A1**). The proof of the statements (ii)-(iii) are also essentially the same as in [12, Proposition 3.14]. We briefly sketch here the main arguments.
(ii) *Weakly narrowly continuity:*
The characteristic function of $\tilde{\mu}_t$ as a probability measure on $Q(A)$ is given by

$$G(t,\xi) = \tilde{\mu}_t(e^{2i\pi\,\mathrm{Re}\,\langle \xi,(A+1)z\rangle_{\mathcal{Z}_0}}).$$

The following inequality holds as in [12, Proposition 3.11] for any $\xi, \xi' \in Q(A)$,

$$|G(t,\xi) - G(t,\xi')| \leqslant \pi \|\xi - \xi'\|_{Q(A)} \int_{\mathcal{Z}_0} \|z\|_{Q(A)}^2\, d\tilde{\mu}_t(z). \tag{5.6.2}$$

Since by Lemma 5.3.5 there exists a time independent constant $C' > 0$ such that $\langle \Psi_t^{(N)}, H_N^0 \Psi_t^{(N)} \rangle \leqslant C'N$, one obtains using Proposition 2.2.22 the uniform estimate,

$$\int_{\mathcal{Z}_0} \|z\|^2_{Q(A)} \, d\tilde{\mu}_t(z) \leqslant C' \,. \tag{5.6.3}$$

Subsequently for any $\xi, \xi' \in Q(A)$,

$$|G(t,\xi) - G(t,\xi')| \lesssim \|\xi - \xi'\|_{Q(A)}. \tag{5.6.4}$$

On the other hand for any $\xi \in Q(A)$ and $t, t' \in \mathbb{R}$, the following estimate holds true

$$|G(t',\xi) - G(t,\xi)| \leqslant \Big| \int_{t'}^{t} \tilde{\mu}_s(e^{2i\pi \operatorname{Re}\langle \xi, (A+1)\, z\rangle} q_1(\xi,s)[z]) \, ds \Big| \leqslant (C'+1) \, |t-t'| \, \|\xi\|_{Q(A)}, \tag{5.6.5}$$

owing to Assumption (**C2**) and Proposition 2.2.22. Now let $g \in S_{cyl}(Q(A))$ based on $\mathfrak{p}Q(A)$ and

$$I_g(t) := \int_{\mathfrak{p}Q(A)} g(z) \; d\tilde{\mu}_t(z) = \int_{\mathfrak{p}Q(A)} \mathcal{F}[g](\xi) \; G(\xi,t) \; L_{\mathfrak{p}}(d\xi).$$

Then we easily check:

- $t \longrightarrow \mathcal{F}[g](\xi) \; G(t,\xi)$ is continuous owing to (5.6.5).
- $\xi \longrightarrow \mathcal{F}[g](\xi) \; G(t,\xi)$ is bounded by a $L_{\mathfrak{p}}(d\xi)$-integrable function.

Thus $I_g(\cdot)$ is continuous for all $g \in S_{cyl}(Q(A))$ and the bound (5.6.3) holds true. Hence we can apply Lemma 5.12-f) in [5] and then conclude that the map $t \to \tilde{\mu}_t$ is weakly narrowly continuous in $Q(A)$. The weakly narrowly continuity on $\mathcal{Z}_0$ follows by a similar argument.

The Liouville equation:
Integrate the expression (5.5.6) with $\mathcal{F}[g](\xi) L_{\wp}(dz)$, hence $\forall t \in \mathbb{R}$, $\forall g \in S_{cyl}(Q(A))$,

$$\partial_t I_g(t) = i \int_{Q(A)} \{q_t; g\}(z) d\tilde{\mu}_t(z),$$

with $q_t(z) = \frac{1}{2} q(z_t^{\otimes 2}, z_t^{\otimes 2})$. Multiplying this expression by $\phi \in C_0^\infty(\mathbb{R})$ and integrating by parts yields

$$\int_{\mathbb{R}} \int_{Q(A)} (\partial_t f(t,z) + i\{q_t(z), f(t,z)\}) \; d\tilde{\mu}_t(z) dt = 0 \,,$$

with $f(t,z) = g(z)\phi(t)$. To conclude, we use the density of $C_0^\infty(\mathbb{R}) \otimes^{alg} C_{0,cyl}^\infty(Q(A))$ in $C_{0,cyl}^\infty(\mathbb{R} \times Q(A))$.
∎

5.6.2 End of the Proof of Theorem 5.2.2

Proof. Assume the hypotheses of Theorem 5.2.2 and consider for a given time $t \in \mathbb{R}$ the family of normal states,

$$\tilde{\varrho}_N(t) = |\widetilde{\Psi}_t^{(N)}\rangle\langle\widetilde{\Psi}_t^{(N)}| = |e^{itH_N^0}e^{-itH_N}\Psi^{(N)}\rangle\langle e^{itH_N^0}e^{-itH_N}\Psi^{(N)}| \,.$$

Suppose that ν is any Wigner measure of $\tilde{\varrho}_N(t)$ then there exists a subsequence $(N_k)_{k\in\mathbb{N}}$ such that $\{\nu\} = \mathcal{M}(\varrho_{N_k}(t), k \in \mathbb{N})$ according to Definition 5.2.1. By Proposition 5.5.2 and 5.6.2, we can extract a subsequence $(N_{k_l})_{l\in\mathbb{N}}$ such that for all $s \in \mathbb{R}$,

$$\mathcal{M}(\tilde{\varrho}_{N_{k_l}}(s), l \in \mathbb{N}) = \{\tilde{\mu}_s\} \qquad \text{with in particular} \qquad \tilde{\mu}_t = \nu \,.$$

We know by Proposition 5.6.2 that $s \in \mathbb{R} \to \tilde{\mu}_s$ solves the Liouville (transport) equation

$$\partial_s\tilde{\mu}_s + i\{q_s(z), \tilde{\mu}_s\} = \partial_s\tilde{\mu}_s + \nabla^T(v_s(z)\tilde{\mu}_s) = 0,$$

in a weak sense, i.e.: For all $f \in C_{0,cyl}^\infty(\mathbb{R}\times\mathcal{Z}_0)$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}}\int_{Q(A)} (\partial_s f(s,z) + i\{q_s(z), f(s,z)\})\, d\tilde{\mu}_s(z)ds \\ &= \int_{\mathbb{R}}\int_{Q(A)} \partial_s f(s,z) + \operatorname{Re}\langle v_s(z), \nabla f(s,z)\rangle_{\mathcal{Z}_0}\, d\tilde{\mu}_s(z)ds \,, \end{aligned}$$

where $v_s(z) = -ie^{isA}[\partial_{\bar{z}}q_0](e^{-isA}z)$, $q_s(z) = \frac{1}{2}q(z_s^{\otimes 2}, z_s^{\otimes 2})$, $z_s = e^{-isA}z$. Here v_s have the interpretation of a velocity vector field and ∇ is the real derivative in $\mathcal{Z}_0$. By Proposition 5.3.5, we see that for any $s \in \mathbb{R}$,

$$\langle\widetilde{\Psi}_s^{(N)}, H_N^0\widetilde{\Psi}_s^{(N)}\rangle \leqslant C'N \,,$$

for some time independent constant $C' > 0$. Thus Proposition 2.2.22 gives for any $s \in \mathbb{R}$,

$$\int_{Q(A)} \|z\|_{Q(A)}^2\, \|z\|_{\mathcal{Z}_0}^2\, d\tilde{\mu}_s(z) \leqslant C' \,.$$

So using assumption (**C2**), for every $T > 0$,

$$\int_0^T\int_{\mathcal{Z}_0} \|v_s(z)\|_{\mathcal{Z}_0}\, d\tilde{\mu}_s(z)\, ds < +\infty. \tag{5.6.6}$$

Now the abstract field equation

$$i\partial_t z = Az + [\partial_{\bar{z}}q_0](z),$$

can be written in the interaction representation as follows:

$$\begin{cases} \partial_t z = v_t(z) = -ie^{itA}[\partial_{\bar{z}}q_0](e^{-itA}z), \\ z_{|t=0} = z_0. \end{cases} \tag{5.6.7}$$

So the above equation (5.6.7) is globally well-posed on $Q(A)$ thanks to the assumption (**C1**). Remember that Proposition 5.6.2 says that the map $s \to \tilde{\mu}_s$ is weakly narrowly continuous on $\mathcal{Z}_0$. Subsequently the measures $(\tilde{\mu}_s)_{s\in\mathbb{R}}$ is satisfying all the assumptions of Theorem 3.1.1. Then we get

$$\forall s \in \mathbb{R},\ \tilde{\mu}_s = \tilde{\Phi}(s,0) * \mu_0,$$

where $\tilde{\Phi}(s,0)$ denotes the well defined flow of the equation (5.6.7). In particular one gets the equality $\nu = \tilde{\Phi}(t,0) * \mu_0$. Since ν is any Wigner measure of $(\tilde{\varrho}_N(t))_{N\in\mathbb{N}}$, one obtains

$$\mathcal{M}(\tilde{\varrho}_N(t), N \in \mathbb{N}) = \{\tilde{\Phi}(t,0) * \mu_0\}\,.$$

Back to the family of normal states,

$$\varrho_N(t) = e^{-itH_N^0}\, \tilde{\varrho}_N(t)\, e^{itH_N^0}\,.$$

We notice that $e^{itH_N^0}\mathcal{W}(\xi)e^{-itH_N^0} = \mathcal{W}(e^{itA}\xi)$ hence a simple computation yields for any $t \in \mathbb{R}$,

$$\mathcal{M}(e^{-itH_N^0}\, \tilde{\varrho}_N(t)\, e^{itH_N^0},\ N \in \mathbb{N}) = \{(e^{-itA})_*\nu, \nu \in \mathcal{M}(\tilde{\varrho}_N(t),\ N \in \mathbb{N})\} = \{(e^{-itA})_*(\tilde{\Phi}(t,0)_*\mu_0)\}.$$

Finally, remark that $\Phi(t,0) = e^{-itA} \circ \tilde{\Phi}(t,0)$. So the main Theorem 5.2.2 is now proved. ∎

5.7 Ground State Energy

In this section we give a proof of the mean field approximation of the ground state energy of trapped many-boson systems (Theorem 5.2.4). Such a result is already proved in a general framework in [77] using a quantum De Finetti theorem. Here the proof comes as a byproduct of general properties of Wigner measures and we presented here as an illustration to our phase-space approach [9, 10, 11, 12]. The proof is based on the key Lemma 5.7.1 below.

Lemma 5.7.1. *Assume* (**A1**)-(**A2**) *and suppose that A has a compact resolvent. Let $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ a sequence of normal states on $\vee^N \mathcal{Z}_0$ satisfying:*

$$\exists C > 0,\ \forall N \in \mathbb{N},\ \langle\Psi^{(N)}, H_N^0\Psi^{(N)}\rangle \leqslant CN. \tag{5.7.1}$$

Then any Wigner measure μ of $\{|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\}_{N\in\mathbb{N}}$ satisfies the equality

$$\mu(S^1_{\mathcal{Z}_0}) = 1,$$

where $S^1_{\mathcal{Z}_0}$ is the unit sphere of the Hilbert space $\mathcal{Z}_0$.

Proof. Without loss of generality we can assume that $\mathcal{M}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|, \mathbb{N} \in N) = \{\mu\}$. Remark that the Wigner measure μ is supported on the unit ball $B(\mathcal{Z}_0)$ owing to Proposition 2.2.22. We shall prove

$$\int_{\mathcal{Z}_0} \|z\|^2_{\mathcal{Z}_0}\, d\mu(z) \geqslant 1. \tag{5.7.2}$$

Indeed if (5.7.2) holds then

$$\int_{\mathcal{Z}_0} 1 - \|z\|^2_{\mathcal{Z}_0}\, d\mu(z) = 0 = \int_{B(\mathcal{Z}_0)} \underbrace{1 - \|z\|^2_{\mathcal{Z}_0}}_{\geqslant 0}\, d\mu(z), \text{ and } \mu(S^1_{\mathcal{Z}_0}) = 1.$$

Since A has a compact resolvent then $A = \sum_{i=0}^{\infty} \lambda_i |e_i\rangle\langle e_i|$, with $(e_i)_{i\geqslant 0}$ is a O.N.B of $\mathcal{Z}_0$, $\lambda_i \geqslant 0$ and $\lim\limits_{i\to+\infty} \lambda_i = +\infty$. Hence if $C(R) := \inf\limits_{i\geqslant R} \lambda_i$ then

$$\lim_{R\to+\infty} C(R) = +\infty.$$

Therefore the following estimate holds true

$$\begin{aligned}\langle \Psi^{(N)}, \sum_{i=1}^{R} |e_i\rangle\langle e_i|\, \Psi^{(N)}\rangle &= 1 - \langle \Psi^{(N)}, \sum_{i=R+1}^{\infty} |e_i\rangle\langle e_i|\, \Psi^{(N)}\rangle = 1 - \langle \Psi^{(N)}, \sum_{i=R+1}^{\infty} \frac{\lambda_i}{C(R)} |e_i\rangle\langle e_i| \Psi^{(N)}\rangle \\ &= 1 - \frac{1}{C(R)} \langle \Psi^{(N)}, A_1 \Psi^{(N)}\rangle \geqslant 1 - \frac{C}{C(R)},\end{aligned}$$

since $\langle \Psi^{(N)}, A_1 \Psi^{(N)}\rangle \leqslant C$ by (5.7.1). Taking the limit $N \to \infty$, we get by Proposition 2.2.21

$$\lim_{N\to\infty} \langle \Psi^{(N)}, b^{Wick}\, \Psi^{(N)}\rangle = \int_{\mathcal{Z}_0} \langle z, \sum_{i=1}^{R} |e_i\rangle\langle e_i|\; z\rangle\, d\mu(z) \geqslant 1 - \frac{C_1}{C(R)}, \tag{5.7.3}$$

where $b(z) = \langle z, \sum_{i=1}^{R} |e_i\rangle\langle e_i|\, z\rangle \in \mathcal{P}_{1,1}(\mathcal{Z}_0)$ and $\tilde{b} = \sum_{i=1}^{R} |e_i\rangle\langle e_i| \in \mathcal{L}^\infty(\mathcal{Z}_0)$. So, we finish the proof by the dominated convergence theorem. ∎

5.7.1 Upper bound

For any $\varphi \in Q(A)$, $\|\varphi\|_{\mathcal{Z}_0} = 1$, take $\Psi^{(N)} = \varphi^{\otimes N} \in Q(H_N^0)$. Compute

$$\begin{aligned}\langle \Psi^{(N)}, H_N\, \Psi^{(N)}\rangle &= \langle \varphi^{\otimes N}, H_N^0 \varphi^{\otimes N}\rangle + \frac{N(N-1)}{2N} q_{1,2}(\varphi^{\otimes N}, \varphi^{\otimes N}) \\ &= N\langle \varphi, A\varphi\rangle + \frac{N(N-1)}{2N} q(\varphi^{\otimes 2}, \varphi^{\otimes 2}).\end{aligned}$$

Hence

$$\frac{E(N)}{N} \leqslant \langle \varphi, A\varphi\rangle + \frac{1}{2} q(\varphi^{\otimes 2}, \varphi^{\otimes 2}) = h(\varphi)\,.$$

Then

$$\liminf_{N\to\infty} \frac{E(N)}{N} \leqslant \inf_{\varphi\in Q(A), \|\varphi\|_{\mathcal{Z}_0}=1} h(\varphi).$$

5.7.2 Lower bound

Let $\{\Psi^{(N)}\}_{N\in\mathbb{N}}$ be a minimizing sequence such that $\Psi^{(N)} \in Q(H_N^0)$, $\|\Psi^{(N)}\|_{\vee^N \mathcal{Z}_0} = 1$ and

$$\frac{1}{N} \langle \Psi^{(N)}, H_N\, \Psi^{(N)}\rangle \leqslant \frac{E(N)}{N} + \frac{1}{N}.$$

Owing to Assumption (**A2**), there exists $C_1 > 0$ such that

$$\frac{1}{N}\langle \Psi^{(N)}, H_N\, \Psi^{(N)}\rangle + C_1 \geqslant 0,$$

and equivalently

$$\frac{1}{N}\langle \Psi^{(N)}, H_N\, \Psi^{(N)}\rangle + C_1 \geqslant \langle \Psi^{(N)}, b(z)^{Wick}_{|\varepsilon=\frac{1}{N}}\, \Psi^{(N)}\rangle \geqslant 0\,,$$

where b is a non-negative monomial on $Q(A)$ given by

$$b(z) = \frac{1}{2}\langle z^{\otimes 2}, A\otimes 1 + 1\otimes A\, z^{\otimes 2}\rangle + \frac{1}{2}q(z^{\otimes 2}, z^{\otimes 2}) + C_1\langle z^{\otimes 2}, z^{\otimes 2}\rangle \in \mathcal{Q}_{2,2}(A)\,.$$

Wick quantization and the classes of symbols $\mathcal{Q}_{p,q}(A)$ are introduced in Section 2.1.4. Now Proposition 2.2.23 yields

$$\liminf_{N\to\infty}\ \frac{E(N)}{N} + C_1 \geqslant \int_{Q(A)} b(z)\; d\mu(z) + C_1 = \int_{Q(A)} h(z)\; d\mu(z) + C_1,$$

where μ is any Wigner measure of $\{|\tilde{\Psi}^{(N)}\rangle\langle\tilde{\Psi}^{(N)}|\}_{N\in\mathbb{N}}$. So using Lemma 5.7.1 we obtain the desired lower bound.

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